Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 4

Fall 2016

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Problem 4.1

Here we shall show that the creation operator, $\hat{a}^\dagger$, does *not* have any non-zero eigenkets. Suppose that a non-zero ket $|\beta\rangle$ satisfies

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle, \tag{1}$$

where β is a complex number. Use the completeness of the number kets to expand $|\beta\rangle$ as follows,

$$|\beta\rangle = \sum_{n=0}^{\infty} b_n|n\rangle,$$

where $b_n = \langle n|\beta\rangle$. Substitute this expansion into Eq. (1) and show that the only possible solution is $b_n = 0$ for all n, i.e., the creation operator has no non-zero eigenkets.

Problem 4.2

Here we shall work out some properties of the coherent states. Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and creation operators for the frequency-ω quantum harmonic oscillator discussed in class. Let $\{\,|\alpha\rangle : \alpha \in \mathcal{C}\,\}$ be the coherent states,

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-|\alpha|^2/2)|n\rangle,$$

where $\{\,|n\rangle : 0 \le n < \infty\,\}$ are the number states and $\alpha \in \mathcal{C}$ is an arbitrary complex number.

(a) Use the orthonormality of the number states, and the power series for the exponential function, to evaluate the inner product $\langle\alpha|\beta\rangle$ between two coherent states $|\alpha\rangle$ and $|\beta\rangle$. Are the coherent states normalized to unit length? Are coherent states with different eigenvalues orthogonal?

(b) Use the completeness of the number states to show that the coherent states are overcomplete, i.e.,

$$\hat{I} = \int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|,$$

where $d^2\alpha \equiv d\alpha_1 d\alpha_2$, with $\alpha_1 \equiv \mathrm{Re}(\alpha)$ and $\alpha_2 \equiv \mathrm{Im}(\alpha)$, and the integration region is the entire complex plane.

(c) Use the result from (b) to show that,

$$\begin{aligned}
\hat{a} &= \hat{a}\hat{I} = \int \frac{d^2\alpha}{\pi}\, \alpha|\alpha\rangle\langle\alpha|, \\
\hat{a}^\dagger &= \hat{I}\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi}\, \alpha^*|\alpha\rangle\langle\alpha|, \\
\hat{a}\hat{a}^\dagger &= \hat{a}\hat{I}\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi}\, |\alpha|^2|\alpha\rangle\langle\alpha|, \\
\hat{a}^\dagger\hat{a} &= \hat{a}\hat{a}^\dagger - \left[\hat{a}, \hat{a}^\dagger\right] = \int \frac{d^2\alpha}{\pi}\, (|\alpha|^2 - 1)|\alpha\rangle\langle\alpha|.
\end{aligned}$$

Problem 4.3

Here we will explore the phase behavior of the quantum harmonic oscillator whose annihilation operator is $\hat{a}$. The Susskind-Glogower phase operator $\widehat{e^{j\phi}}$ associated with $\hat{a}$ is defined as follows

$$\widehat{e^{j\phi}} \equiv (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}.$$

(Note that the "widehat" symbol is used to indicate that this is *not* the exponentiation of j times an Hermitian operator $\hat{\phi}$.)

(a) Find the number-ket representation of $\widehat{e^{j\phi}}$, i.e., find c_{nm} such that

$$\widehat{e^{j\phi}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}|n\rangle\langle m|,$$

where $|n\rangle$ and $\langle m|$ are number states.

Useful Fact: If $\hat{B}$ is an Hermitian operator with the eigenvalue-eigenket decomposition

$$\hat{B} = \sum_n b_n|b_n\rangle\langle b_n|,$$

and if $F(\cdot)$ is a deterministic function, then

$$F(\hat{B}) = \sum_n F(b_n)|b_n\rangle\langle b_n|$$

(b) Show that

$$|e^{j\phi}\rangle \equiv \sum_{n=0}^{\infty} e^{jn\phi}|n\rangle,$$

where $|n\rangle$ is the number state, is an eigenket of $\widehat{e^{j\phi}}$, and determine its associated eigenvalue.

(c) Show that $\{\, |e^{j\phi}\rangle : -\pi \le \phi \le \pi \,\}$ resolves the identity, i.e., prove that

$$\hat{I} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, |e^{j\phi}\rangle\langle e^{j\phi}|.$$

It follows from this result that

$$\hat{\Pi}(\phi) \equiv \frac{|e^{j\phi}\rangle\langle e^{j\phi}|}{2\pi}$$

is a probability operator-valued measurement (POVM) for the phase of the $\hat{a}$ mode.

Problem 4.4

Here we shall develop a little commutator calculus that will be needed in the next problem. Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and creation operators, respectively, of a quantum harmonic oscillator, and let $\hat{a}_1 \equiv \mathrm{Re}(\hat{a})$ and $\hat{a}_2 \equiv \mathrm{Im}(\hat{a})$ be the associated quadrature operators, i.e., the normalized versions of position and momentum for a mechanical oscillator, or charge and flux for an LC oscillator.

(a) Use $[\hat{a}_1, \hat{a}_2] = j/2$ to show that

$$\left[\hat{a}_1, \hat{a}_2^2\right] = j\hat{a}_2.$$

Assume that

$$\left[\hat{a}_1, \hat{a}_2^k\right] = jk\hat{a}_2^{k-1}/2, \quad \text{for } k > 2.$$

Show that

$$\left[\hat{a}_1, \hat{a}_2^{k+1}\right] = j(k+1)\hat{a}_2^k/2,$$

thus completing the induction proof that

$$\left[\hat{a}_1, \hat{a}_2^k\right] = jk\hat{a}_2^{k-1}/2, \quad \text{for } k = 1, 2, 3, \ldots$$

By analogy with classical functions we *define* the following operator derivative,

$$\frac{d\hat{a}_2^k}{d\hat{a}_2} \equiv k\hat{a}_2^{k-1},$$

so that

$$\left[\hat{a}_1, \hat{a}_2^k\right] = (j/2)\frac{d\hat{a}_2^k}{d\hat{a}_2}, \quad \text{for } k = 1, 2, 3, \ldots$$

(b) Follow a similar induction argument to that used in (a) to prove the commutation rule,

$$\left[\hat{a}_2, \hat{a}_1^k\right] = -jk\hat{a}_1^{k-1}/2 = -(j/2)\frac{d\hat{a}_1^k}{d\hat{a}_1}, \quad \text{for } k = 1, 2, 3, \ldots,$$

where the last equality *defines* the operator derivative.

(c) Suppose that $F(\alpha_1)$ and $G(\alpha_2)$ are functions of real variables α_1 and α_2 that have convergent Taylor's series,

$$\begin{aligned} F(\alpha_1) &= \sum_{n=0}^{\infty} \frac{\alpha_1^n}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0}, \quad \text{for } -\infty < \alpha_1 < \infty, \\ G(\alpha_2) &= \sum_{n=0}^{\infty} \frac{\alpha_2^n}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0}, \quad \text{for } -\infty < \alpha_2 < \infty. \end{aligned}$$

Define the operators $F(\hat{a}_1)$ and $G(\hat{a}_2)$ by the operator-valued Taylor's series,

$$\begin{aligned} F(\hat{a}_1) &= \sum_{n=0}^{\infty} \frac{\hat{a}_1^n}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0}, \\ G(\hat{a}_2) &= \sum_{n=0}^{\infty} \frac{\hat{a}_2^n}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0}. \end{aligned}$$

Use the results of (a) and (b) to find the commutators $[\hat{a}_1, G(\hat{a}_2)]$ and $[\hat{a}_2, F(\hat{a}_1)]$.

Problem 4.5

Here we shall show that the eigenkets of a quadrature operator can be found from a translation operator applied to the zero-eigenvalue eigenket.

(a) Assume that $|\alpha_1\rangle_1$ is an eigenket of the quadrature operator $\hat{a}_1$ with eigenvalue α_1. Because $\hat{a}_1$ is Hermitian, α_1 is a real number. Define a translation operator,

$$\hat{A}_1(\xi) \equiv \exp(-2j\xi\hat{a}_2) = \sum_{n=0}^{\infty} \frac{(-2j\xi)^n}{n!}\hat{a}_2^n, \quad \text{for } -\infty < \xi < \infty.$$

Use

$$\hat{a}_1\hat{A}_1(\xi)|\alpha_1\rangle_1 = \hat{A}_1(\xi)\hat{a}_1|\alpha_1\rangle_1 + \left[\hat{a}_1, \hat{A}_1(\xi)\right]|\alpha_1\rangle_1,$$

and the results from Problem 4.3 to show that $\hat{A}_1(\xi)|\alpha_1\rangle_1$ is an eigenket of $\hat{a}_1$ with eigenvalue $\alpha_1 + \xi$, for any real number ξ.

(b) Let $|0\rangle_1$ be the $\hat{a}_1$ eigenket whose eigenvalue is zero. Show that

$$|\alpha_1\rangle_1 = \exp(-2j\alpha_1\hat{a}_2)|0\rangle_1,$$

is an $\hat{a}_1$ eigenket with eigenvalue α_1 and that ${}_1\langle\alpha_1|\alpha_1\rangle_1 = {}_1\langle 0|0\rangle_1$.

(c) Assume that $|\alpha_2\rangle_2$ is an eigenket of the quadrature operator $\hat{a}_2$ with eigenvalue α_2. Because $\hat{a}_2$ is Hermitian, α_2 is a real number. Define a translation operator,

$$\hat{A}_2(\xi) \equiv \exp(2j\xi\hat{a}_1) = \sum_{n=0}^{\infty} \frac{(2j\xi)^n}{n!}\hat{a}_1^n, \quad \text{for } -\infty < \xi < \infty.$$

Use

$$\hat{a}_2 \hat{A}_2(\xi)|\alpha_2\rangle_2 = \hat{A}_2(\xi)\hat{a}_2|\alpha_2\rangle_2 + \left[\hat{a}_2, \hat{A}_2(\xi)\right]|\alpha_2\rangle_2,$$

and the results from Problem 4.3 to show that $\hat{A}_2(\xi)|\alpha_2\rangle_2$ is an eigenket of $\hat{a}_2$ with eigenvalue $\alpha_2 + \xi$, for any real number ξ.

(d) Let $|0\rangle_2$ be the $\hat{a}_2$ eigenket whose eigenvalue is zero. Show that

$$|\alpha_2\rangle_2 = \exp(2j\alpha_2\hat{a}_1)|0\rangle_2,$$

is an $\hat{a}_2$ eigenket with eigenvalue α_2 and that ${}_2\langle\alpha_2|\alpha_2\rangle_2 = {}_2\langle 0|0\rangle_2$.

Problem 4.6

Here we shall continue our development of the quadrature-operator eigenkets. The results of Problem 4.5 show that these operators have continuous spectra, i.e., their eigenvalues are $\{-\infty < \alpha_1 < \infty\}$ and $\{-\infty < \alpha_2 < \infty\}$, respectively. Because $\hat{a}_1$ and $\hat{a}_2$ are observables, the appropriate orthonormality and completeness conditions for their eigenkets are therefore,

$$\begin{aligned} {}_1\langle\alpha_1'|\alpha_1\rangle_1 &= \delta(\alpha_1 - \alpha_1') \quad \text{and} \quad {}_2\langle\alpha_2'|\alpha_2\rangle_2 = \delta(\alpha_2 - \alpha_2'), \\ \hat{I} &= \int_{-\infty}^{\infty} d\alpha_1\, |\alpha_1\rangle_{11}\langle\alpha_1| = \int_{-\infty}^{\infty} d\alpha_2\, |\alpha_2\rangle_{22}\langle\alpha_2|. \end{aligned}$$

(a) Use the Heisenberg uncertainty principle to show that $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$ have infinite average energy, i.e., that $\langle\hat{H}\rangle = \hbar\omega(\langle\hat{a}_1^2\rangle + \langle\hat{a}_2^2\rangle) = \infty$ for these states.

(b) We want to determine the relationship between the eigenkets $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$. Use the results of Problem 4.5 to show that

$${}_2\langle\alpha_2|\alpha_1\rangle_1 = \exp(-2j\alpha_1\alpha_2)\,{}_2\langle 0|0\rangle_1.$$

Hint: The power series expansion of $\hat{A}_1(\xi)$ can be used to show that $|\alpha_2\rangle_2$ is an eigenket of this translation operator; likewise $|\alpha_1\rangle_1$ is an eigenket of the translation operator $\hat{A}_2(\xi)$.

(c) Find $|{}_2\langle 0|0\rangle_1|^2$ by evaluating

$${}_2\langle\alpha_2'|\alpha_2\rangle_2 = {}_2\langle\alpha_2'|\hat{I}|\alpha_2\rangle_2 = {}_2\langle\alpha_2'|\left(\int_{-\infty}^{\infty} d\alpha_1\, |\alpha_1\rangle_{11}\langle\alpha_1|\right)|\alpha_2\rangle_2,$$

using the result of (b). Assume that ${}_2\langle 0|0\rangle_1$ is positive real to completely pin down ${}_2\langle\alpha_2|\alpha_1\rangle_1$.