Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 QUANTUM OPTICAL COMMUNICATION

Problem Set 5
Fall 2016

Issued: Thursday, October 6, 2016   **Due:** Thursday, October 13, 2016

Reading: For normally-ordered forms:

- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) Sect. 3.2.

Problem 5.1
Here we shall derive the signal-to-noise ratio (SNR) optimality of squeezed states for quadarature measurements. Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and creation operators, respectively, of a quantum harmonic oscillator, and let $\hat{a}_1 \equiv \mathrm{Re}(\hat{a})$ and $\hat{a}_2 \equiv \mathrm{Im}(\hat{a})$ be the associated quadrature operators. We want to find the state $|\psi\rangle$ that maximizes the SNR of the $\hat{a}_1$ measurement,

$$\mathrm{SNR} \equiv \frac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle},$$

when the mean of the quadrature measurement must be positive $\langle \hat{a}_1 \rangle > 0$, and the state must satisfy the average photon-number constraint,

$$\langle \hat{a}^\dagger \hat{a} \rangle \leq N.$$

(a) Express $\langle \hat{a}^\dagger \hat{a} \rangle$ in terms of the squared-means and the variances of the $\hat{a}_1$ and $\hat{a}_2$ measurements. Use this result to argue that the optimum state should have $\langle \hat{a}_2 \rangle = 0$ and $\langle \hat{a}^\dagger \hat{a} \rangle = N$, and thus satisfy

$$\mathrm{SNR} = \frac{N + 1/2 - \langle \Delta \hat{a}_2^2 \rangle}{\langle \Delta \hat{a}_1^2 \rangle} - 1.$$

(b) Use the result of (a) to show that the optimum state must be a minimum uncertainty product state for the Heisenberg inequality $\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \geq 1/16$. Optimize your resulting SNR expression over $0 \leq \langle \Delta \hat{a}_1^2 \rangle$.

(c) Show that your optimum SNR expression from (b) is achieved by the squeezed state $|\beta; \mu, \nu\rangle$, where $\beta = \sqrt{N(N+1)}$, $\mu = (N+1)/\sqrt{2N+1}$, and $\nu = N/\sqrt{2N+1}$.

(d) Compare the SNR achieved by your optimum squeezed state from (c) with that of the coherent state, $|\sqrt{N}\rangle$, of the same average photon number.

Problem 5.2

Here we shall introduce the notion of normally-ordered forms. Consider a quantum harmonic oscillator with annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$. Operators built up from Taylor's series of the form,

$$F(\hat{a}^\dagger, \hat{a}) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{nm} \hat{a}^{\dagger n} \hat{a}^m,$$

are said to be in normal order, because (in each nm-term) all the creation operators stand to the left of all the annihilation operators. On the other hand, operators built up from Taylor's series of the form,

$$G(\hat{a}, \hat{a}^\dagger) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{nm} \hat{a}^n \hat{a}^{\dagger m},$$

are said to be in anti-normal order, because (in each nm-term) all the annihilation operators stand to the left of all the creation operators. By (repeated) use of the $[\hat{a}, \hat{a}^\dagger]$ commutator it is possible to convert a normally-ordered operator into an equivalent anti-normally ordered operator. Normal order is very convenient, as we shall see, when calculations are performed using coherent states

(a) Find the normally-ordered form, $F^{(n)}(\hat{a}^\dagger, \hat{a})$, and the anti-normally ordered form, $F^{(a)}(\hat{a}, \hat{a}^\dagger)$, of the operator $\hat{F} \equiv \hat{a}\hat{a}^\dagger\hat{a}$. Find $\langle \alpha|\hat{F}|\alpha \rangle$, where $|\alpha\rangle$ is a coherent state, and verify that it satisfies,

$$\langle \alpha|\hat{F}|\alpha \rangle = F^{(n)}(\alpha^*, \alpha),$$

i.e., it equals the normally-ordered form of $\hat{F}$ with the classical complex numbers α^* and α replacing the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, respectively.

(b) Use the fact that the coherent states resolve the identity,

$$\hat{I} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|,$$

to show that any operator $\hat{G}$ is completely determined by its coherent-state matrix elements, $\langle \alpha|\hat{G}|\beta \rangle$, via,

$$\hat{G} = \int\int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \langle \alpha|\hat{G}|\beta \rangle |\alpha\rangle\langle\beta|.$$

(c) Suppose that we regard $F^{(n)}(\alpha^*, \alpha) = \langle \alpha|F^{(n)}(\hat{a}^\dagger, \hat{a})|\alpha \rangle$ from (a) to be a determinstic function of two *independent* classical arguments, α^* and α. Show that,

$$\langle \alpha|\hat{F}|\beta \rangle = F^{(n)}(\alpha^*, \beta)\langle\alpha|\beta\rangle, = F^{(n)}(\alpha^*, \beta)e^{-(|\alpha|^2+|\beta|^2)/2+\alpha^*\beta},$$

for any two coherent states $|\alpha\rangle$ and $|\beta\rangle$. In conjunction with (b), this implies that $\hat{F}$ is completely determined by its *diagonal* elements in the coherent-state expansion. Moreover, these diagonal elements are immediately available from the normally-ordered form of $\hat{F}$ (and vice versa).

(d) Of particular interest for quantum optics work is the normally-ordered representation for the density operator, $\hat{\rho}$, which specifies the state (pure or mixed) of the oscillator. Show that $\rho^{(n)}(\alpha^*, \alpha) \equiv \langle\alpha|\hat{\rho}|\alpha\rangle$ satisfies the following conditions:

$$\rho^{(n)}(\alpha^*, \alpha) \geq 0, \quad \text{for all } \alpha,$$

$$\int \frac{d^2\alpha}{\pi}\, \rho^{(n)}(\alpha^*, \alpha) = 1.$$

Thus, $p(\alpha_1, \alpha_2) \equiv \rho^{(n)}(\alpha^*, \alpha)/\pi$, where $\alpha \equiv \alpha_1 + j\alpha_2$, is a possible classical joint-probability density for two real-valued random variables.

Problem 5.3

In classical probability theory, probability densities and characteristic functions are Fourier transform pairs, and some calculations are easier to perform in one domain than the other. In quantum mechanics, the density operator takes the place of the probability density and, because of commutation rules, there are several different characteristic functions that can be defined. Here we will introduce the three most important of these characteristic functions. Consider a quantum harmonic oscillator with annihilation operator $\hat{a}$, creation operator $\hat{a}^\dagger$, and density operator $\hat{\rho}$. The anti-normally ordered characteristic function is defined to be,

$$\chi_A^\rho(\zeta^*, \zeta) \equiv \mathrm{tr}\left(\hat{\rho}e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger}\right),$$

the normally-ordered chacteristic function is defined to be,

$$\chi_N^\rho(\zeta^*, \zeta) \equiv \mathrm{tr}\left(\hat{\rho}e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}\right),$$

and the Wigner characteristic function is defined to be,

$$\chi_W^\rho(\zeta^*, \zeta) \equiv \mathrm{tr}\left(\hat{\rho}e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger}\right),$$

In these expressions, ζ is a complex number whose real and imaginary parts are ζ_1 and ζ_2, respectively.

(a) Let $\hat{A}$ and $\hat{B}$ be non-commuting operators that commute with their commutator, i.e.,

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0.$$

It can be shown that

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]/2}.$$

Use this result to relate the three characteristic functions to one another. Find all three characteristic functions for the pure-state density operator $\hat{\rho} = |\alpha\rangle\langle\alpha|$, where $|\alpha\rangle$ is a coherent state.

(b) Let $\rho^{(n)}(\alpha^*, \alpha) \equiv \langle\alpha|\hat{\rho}|\alpha\rangle$ be the diagonal matrix elements of the density operator in the coherent-state basis. Show that,

$$\rho^{(n)}(\alpha^*, \alpha) = \int \frac{d^2\zeta}{\pi} \chi_A^\rho(\zeta^*, \zeta) e^{-\zeta\alpha^* + \zeta^*\alpha}.$$

(c) Use the fact that $\hat{\rho}$ is determined by its diagonal elements in the coherent-state basis to prove that

$$\hat{\rho} = \int \frac{d^2\zeta}{\pi} \chi_A^\rho(\zeta^*, \zeta) e^{-\zeta\hat{a}^\dagger} e^{\zeta^*\hat{a}}.$$

(d) Suppose that $\hat{\rho}$ has a proper P-representation, i.e., $\hat{\rho}$ satisfies,

$$\hat{\rho} = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

where $|\alpha\rangle$ is the coherent state, and $P(\alpha, \alpha^*)$ is a classical joint probability density for α_1 and α_2, the real and imaginary parts of α. Show that

$$P(\alpha, \alpha^*) = \int \frac{d^2\zeta}{\pi^2} \chi_N^\rho(\zeta^*, \zeta) e^{-\zeta\alpha^* + \zeta^*\alpha}.$$

(e) Suppose that the θ-quadrature of the oscillator is measured, i.e., we measure the observable $\hat{a}_\theta \equiv \mathrm{Re}(\hat{a}e^{-j\theta}) = \hat{a}_1\cos(\theta) + \hat{a}_2\sin(\theta)$. The outcome of this measurement is a real-valued classical random variable α_θ, whose characteristic function satisfies,

$$M_{\alpha_\theta}(jv) \equiv E(e^{jv\alpha_\theta}) = \mathrm{tr}\left(\hat{\rho}e^{jv\hat{a}_\theta}\right).$$

Find $M_{\alpha_\theta}(jv)$ from the Wigner characteristic function.

Problem 5.4
Here we shall show that it is easy to calculate number-operator and quadrature-operator measurement statistics when the oscillator has a proper P-representation.

(a) Consider a quantum harmonic oscillator whose density operator $\hat{\rho}$ has a proper P-representation, $P(\alpha, \alpha^*)$. Show that we can regard the oscillator as being in a mixed state in which the coherent state $|\alpha\rangle$ occurs with probability density $P(\alpha, \alpha^*)$.

(b) Use the result of (a) to show that the probability that a number-operator measurement yields outcome n satisfies,

$$\Pr(\hat{N} \text{ outcome } = n) = \int d^2\alpha\, P(\alpha, \alpha^*) \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \dots$$

Show that the variance of the $\hat{N}$ measurement for this state always equals or exceeds its mean value. Use this result to show that the density operator $\hat{\rho} = |n\rangle\langle n|$, where $|n\rangle$ is the number ket, does not have a proper P-representation for $n > 0$.

(c) Use the result of (a) to show that the probability density for the $\hat{a}_1$ measurement to yield outcome a_1 is

$$p(\hat{a}_1 \text{ outcome } = a_1) = \int d^2\alpha\, P(\alpha, \alpha^*) \frac{\exp[-2(a_1 - \alpha_1)^2]}{\sqrt{\pi/2}}.$$

Show that the variance of the $\hat{a}_1$ measurement for this state always equals or exceeds $1/4$. Use this result to show that the squeezed state $|\beta; \mu, \nu\rangle$, with $\mu, \nu > 0$, does not have a proper P-representation.

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