Green's functions for planarly layered media

Massachusetts Institute of Technology
6.635 lecture notes

1 Introduction: Green's functions

The Green's functions is the solution of the wave equation for a point source (dipole). For scalar problems, the wave equation is written as ($k_0 = \omega\sqrt{\epsilon\mu}$):

$$(\nabla^2 + k_0^2)\, g(\bar{r}, \bar{r}') = -\delta(\bar{r} - \bar{r}')\,, \tag{1}$$

and the solution for an unbounded medium is:

$$g(\bar{r}, \bar{r}') = \frac{e^{ik_0|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|}\,. \tag{2}$$

From Maxwell's equations in frequency domain with an $e^{i\omega t}$ dependency, the wave equation for electric field $\bar{E}(\bar{r})$ is:

$$\begin{aligned} \nabla \times \nabla \times \bar{E}(\bar{r}) - k_0^2 \bar{E}(\bar{r}) &= i\omega \bar{J}(\bar{r})\,, \\ &= 0 \qquad \text{for source free case}\,. \end{aligned} \tag{3}$$

Therefore, the free-space dyadic Green's function satisfies

$$\nabla \times \nabla \times \overline{\overline{G}}(\bar{r}, \bar{r}') - k_0^2 \overline{\overline{G}}(\bar{r}, \bar{r}') = \overline{\overline{I}}\, \delta(\bar{r} - \bar{r}')\,, \tag{4}$$

which solution is

$$\overline{\overline{G}}(\bar{r}, \bar{r}') = \left(\overline{\overline{I}} + \frac{1}{k_0^2}\nabla\nabla\right) g(\bar{r}, \bar{r}')\,, \tag{5}$$

(check using $\nabla \times \nabla \times (\overline{\overline{I}} g) = \nabla\nabla g - \nabla \cdot (\nabla g)\overline{\overline{I}}$ and Eq. (1)).

For the use of Green's functions in scattering problems, it is useful to express the Green's function in the same coordinates as the problem, which can be rectangular, cylindrical, spherical, etc. Here we shall concentrate on the rectangular representation (Cartesian).

2 Cartesian coordinates

2.1 Scalar Green's function

The formulae are derived from Eq. (1) and the Fourier transform of the quantities:

$$g(\bar{r},\bar{r}') = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, g(\bar{k})\,, \tag{6a}$$

$$\delta(\bar{r}-\bar{r}') = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\,, \tag{6b}$$

where $\bar{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ and $\mathrm{d}\bar{k} = \mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z$.

Upon using Eq. (1), we write:

$$(\nabla^2 + k_0^2) \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, g(\bar{k}) = -\iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\,, \tag{7}$$

Introducing the differential operator ($\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$) we write:

$$\begin{aligned}(\nabla^2 + k_0^2) \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, g(\bar{k}) &= \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; (\nabla^2 + k_0^2) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, g(\bar{k}) \\ &= \iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; (-k_x^2 - k_y^2 - k_z^2 + k_0^2) e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\, g(\bar{k}) \\ &= -\iiint_{-\infty}^{+\infty} \mathrm{d}\bar{k}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\,, \end{aligned} \tag{8}$$

from which we conclude that

$$g(\bar{k}) = \frac{1}{k_x^2 + k_y^2 + k_z^2 - k_0^2}\,. \tag{9}$$

Using Eq. (6a), we therefore need to evaluate the following integral:

$$g(\bar{r},\bar{r}') = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z\; \frac{1}{k_x^2 + k_y^2 + k_z^2 - k_0^2}\; e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\,. \tag{10}$$

Note that Eq. (10) can be integrated along one of the three axis. In remote sensing application, the vertical axis is usually taken to be the z axis, (xy) being the transverse plane (planar components). We therefore choose to evaluate Eq. (10) along k_z, and we split:

$$\bar{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = \bar{k}_\perp + k_z\hat{z}\,, \tag{11a}$$

$$\bar{r} = \bar{r}_\perp + z\hat{z}\,, \tag{11b}$$

$$\bar{r}' = \bar{r}'_\perp + z'\hat{z}\,. \tag{11c}$$

We will perform the integral of Eq. (10) in the complex plane, using Cauchy's theorem and the Residue theorem. Before doing this, we have to be careful not to have divergent integrals. Since we integrate in k_z, the condition is:

$$\lim_{k_z \to \infty} e^{ik_z z} < +\infty \,. \tag{12}$$

If we write k_z as $k_z = k'_z + ik''_z$ ($k'_z \in \mathbf{R}$, $k''_z \in \mathbf{R}$), we see that

- if $z > 0$, we have to choose $k''_z > 0$, which means that for complex plane integration, we need to deform the contour into the upper plane.
- if $z < 0$, we have to choose $k''_z < 0$, which corresponds to a deformation into the lower plane.

In addition, we see from Eq. (10) that the integrand has a pole at

$$k_{0_z}^2 = k_0^2 - k_x^2 - k_y^2 = k_0^2 - k_\perp^2 \,. \tag{13}$$

We therefore need to evaluate Eq. (10) via the Residue theorem.

Calculus: let us just write the integral in dk_z for $z > 0$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{k_z^2 - k_{0_z}^2} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')} &= 2i\pi \, \mathrm{Res}\left[\frac{1}{k_z^2 - k_{0_z}^2} e^{i\bar{k}\cdot(\bar{r}-\bar{r}')}\right] \\
&= 2i\pi \lim_{k_z \to k_{0_z}} \frac{k_z - k_{0_z}}{(k_z - k_{0_z})(k_z + k_{0_z})} e^{i\bar{k}_\perp\cdot(\bar{r}_\perp - \bar{r}'_\perp)} \, e^{ik_z(z-z')} \\
&= 2i\pi \frac{1}{2k_{0_z}} e^{i\bar{k}_\perp\cdot(\bar{r}_\perp - \bar{r}'_\perp)} \, e^{ik_{0_z}(z-z')} \,,
\end{aligned} \tag{14}$$

so that

$$g(\bar{r}, \bar{r}') = \frac{i}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}\bar{k}_\perp \, \frac{1}{2k_{0_z}} \, e^{i\bar{k}_\perp\cdot(\bar{r}_\perp - \bar{r}'_\perp)} \, e^{ik_{0_z}(z-z')} \,, \qquad \text{for } z - z' > 0. \tag{15}$$

The treatment for $z < 0$ follows the same reasoning so that we write for all $(z - z')$:

$$\forall (z - z') \in \mathbf{R}, \qquad g(\bar{r}, \bar{r}') = \frac{i}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}\bar{k}_\perp \, \frac{1}{2k_{0_z}} \, e^{i\bar{k}_\perp\cdot(\bar{r}_\perp - \bar{r}'_\perp)} \, e^{ik_{0_z}|z-z'|} \,. \tag{16}$$

2.2 Dyadic Green's function

From Eq. (16) and Eq. (26), we can get the dyadic Green's functions.

Note that $\nabla\nabla$ is a dyadic operator (give a dyad when applied to a scalar) and can be exchanged with the integral sign. In addition, it only applies to the exponential terms so that we actually need to evaluate:

$$\nabla\nabla \left[e^{i\bar{k}_\perp\cdot(\bar{r}_\perp - \bar{r}'_\perp)} \, e^{ik_{0_z}|z-z'|} \right] \tag{17}$$

or, by a simple change of variables:

$$\nabla\nabla\left[e^{i\bar{k}_\perp\cdot\bar{r}_\perp}\,e^{ik_{0_z}|z|}\right] \tag{18}$$

Calculus: Let us first consider $z>0$ and

$$\nabla\nabla(e^{i\bar{k}_\perp\cdot\bar{r}_\perp}\,e^{ik_{0_z}|z|}) = \nabla\nabla\, f(x,y,z)\,. \tag{19}$$

Various derivatives will be:

- $\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y,z) = -k_x^2\, f(x,y,z)\,,$
- $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y,z) = -k_x k_y\, f(x,y,z)\,,$
- ...

and identically for $z<0$. At $z=0$ however,

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\,e^{ik_{0_z}|z|} &= \frac{\partial}{\partial z}\left\{ik_{0_z}\,e^{ik_{0_z}|z|}\frac{\partial}{\partial z}|z|\right\}\\ &= ik_{0_z}\left\{ik_{0_z}\,e^{ik_{0_z}|z|}\left(\frac{\partial}{\partial z}|z|\right)^2 + ik_{0_z}\,e^{ik_{0_z}|z|}\frac{\partial^2}{\partial z^2}|z|\right\}\\ &= 2ik_{0_z}\delta(z) - k_{0_z}^2\,e^{ik_{0_z}|z|}\,.\end{aligned} \tag{20}$$

Using these results, we write:

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\,g(\bar{r}) &= \frac{i}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,\frac{1}{2k_{0_z}}\,e^{i\bar{k}_\perp\cdot\bar{r}_\perp}\left[2ik_{0_z}\delta(z) - k_{0_z}^2\,e^{ik_{0_z}|z|}\right]\\ &= -\frac{\delta(z)}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,e^{i\bar{k}_\perp\cdot\bar{r}_\perp} - \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,k_{0_z}\,e^{i\bar{k}_\perp\cdot\bar{r}_\perp}\,e^{ik_{0_z}|z|}\\ &= -\delta(\bar{r}) - \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,k_{0_z}\,e^{i\bar{k}_\perp\cdot\bar{r}_\perp}\,e^{ik_{0_z}|z|}\,.\end{aligned} \tag{21}$$

Again, all the other terms of $\nabla\nabla$ applied to the integrand give $-\bar{k}\bar{k}$ so that the Green's function becomes:

$$\overline{\overline{G}}(\bar{r},\bar{r}') = -\hat{z}\hat{z}\frac{\delta(\bar{r})}{k_0^2} + \frac{i}{8\pi^2}\begin{cases}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\left[\overline{\overline{I}} - \frac{\bar{k}\bar{k}}{k_0^2}\right]e^{i\bar{k}\cdot\bar{r}} & \text{for } z>0\,,\\ \iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\left[\overline{\overline{I}} - \frac{\bar{K}\bar{K}}{k_0^2}\right]e^{i\bar{K}\cdot\bar{r}} & \text{for } z<0\,,\end{cases} \tag{22}$$

where

$$\bar{k} = k_x\hat{x} + k_y\hat{y} + k_{0_z}\hat{z}\,, \tag{23a}$$

$$\bar{K} = k_x\hat{x} + k_y\hat{y} - k_{0_z}\hat{z}\,. \tag{23b}$$

Some notes:

1. The Dirac delta function is known as the singularity of the Green's function and is important in calculating the fields in the source region.

2. The different signs ensure that the integral converges for evanescent waves, *i.e.* when $k_x^2 + k_y^2 > k_0^2$.

3. The square bracket in the expression of the Green's functions can be expressed in terms of superposition of TE and TM waves, as we shall see.

2.3 Superposition of TE and TM waves

Based on $\bar{k}$, we can form an orthonormal system for TE/TM polarized waves:

$$\text{TE:}\quad \hat{e}(k_{0_z}) = \frac{\bar{k} \times \hat{z}}{|\bar{k} \times \hat{z}|} = \frac{1}{\sqrt{k_x^2 + k_y^2}}[\hat{x}k_y - \hat{y}k_x] = \frac{1}{k_\rho}(\hat{x}k_y - \hat{y}k_x)\,, \tag{24a}$$

$$\text{TM:}\quad \hat{h}(k_{0_z}) = \frac{1}{k_0}\,\hat{e}(k_{0_z}) \times \bar{k} = -\frac{k_{0_z}}{k_o k_\rho}\,(\hat{x}k_y + \hat{y}k_x) + \frac{k_\rho}{k_o}\hat{z}\,. \tag{24b}$$

The three vectors $\hat{k}$, $\hat{h}$ and $\hat{e}$ form an orthonormal system, in which:

$$\overline{\overline{I}} = \hat{k}\hat{k} + \hat{e}(k_{0_z})\hat{e}(k_{0_z}) + \hat{h}(k_{0_z})\hat{h}(k_{0_z})\,. \tag{25}$$

After translating to the origin, we get for the dyadic Green's functions:

$$\overline{\overline{G}}(\bar{r}, \bar{r}') = -\hat{z}\hat{z}\frac{\delta(\bar{r})}{k_0^2} + \frac{i}{8\pi^2}\begin{cases} \displaystyle\iint_{-\infty}^{\infty} \mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\left[\hat{e}(k_{0_z})\hat{e}(k_{0_z}) + \hat{h}(k_{0_z})\hat{h}(k_{0_z})\right] e^{i\bar{k}\cdot(\bar{r}-\bar{r}')} & \text{for } z > z'\,, \\ \displaystyle\iint_{-\infty}^{\infty} \mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\left[\hat{e}(-k_{0_z})\hat{e}(-k_{0_z}) + \hat{h}(-k_{0_z})\hat{h}(-k_{0_z})\right] e^{i\bar{K}\cdot(\bar{r}-\bar{r}')} & \text{for } z < z'\,, \end{cases} \tag{26}$$

where

$$\hat{e}(-k_{0_z}) = \hat{e}(k_{0_z})\,, \tag{27a}$$

$$\hat{h}(-k_{0_z}) = \frac{\hat{e}(-k_{0_z}) \times \bar{K}}{k_0}\,. \tag{27b}$$

(Note that $\bar{K}$, $\hat{e}(-k_{0_z})$ and $\hat{h}(-k_{0_z})$ form another orthonormal set of vectors about $\bar{K}$).

2.4 Treatment of layered media

Depending upon the medium under study and the location of the source, the kernel of Eq. (26) will have to be modified. To make it more clear, we can gather the terms in the Green's function relative to the source (primed coordinates) and those relative to the source.

$$\overline{\overline{G}}(\bar{r},\bar{r}')=-\hat{z}\hat{z}\frac{\delta(\bar{r})}{k_0^2}+\frac{i}{8\pi^2}\begin{cases}\iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\{[\hat{e}(k_{0_z})\,e^{i\bar{k}\cdot\bar{r}}]\,\hat{e}(k_{0_z})\,e^{-i\bar{k}\bar{r}'}+[\hat{h}(k_{0_z})\,e^{i\bar{k}\cdot\bar{r}}]\,\hat{h}(k_{0_z})\,e^{-i\bar{k}\bar{r}'}\} & \text{for } z>z'\,,\\ \iint_{-\infty}^{\infty}\mathrm{d}\bar{k}_\perp\,\frac{1}{k_{0_z}}\{[\hat{e}(-k_{0_z})\,e^{i\bar{K}\cdot\bar{r}}]\,\hat{e}(-k_{0_z})\,e^{-i\bar{K}\bar{r}'}+[\hat{h}(-k_{0_z})\,e^{i\bar{K}\cdot\bar{r}}]\,\hat{h}(-k_{0_z})\,e^{-i\bar{K}\bar{r}'}\} & \text{for } z<z'\,,\end{cases} \tag{28}$$

If we now consider a layered medium problem, with an arbitrary number of layers and a source in the top region (incident wave), we write:

$$\overline{\overline{G}}(\bar{r},\bar{r}')_{i0}=\frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\mathrm{d}k_x\mathrm{d}k_y\,\frac{1}{k_{0_z}}\left\{\bar{K}_e\,\hat{e}(-k_{0_z})\,e^{-i\bar{K}\cdot\bar{r}'}+\bar{K}_h\,\hat{h}(-k_{0_z})\,e^{-i\bar{K}\cdot\bar{r}'}\right\} \tag{29}$$

with

1. For $z<z'$, $i=0$:

$$\bar{K}_e = \hat{e}(-k_{0_z})\,e^{i\bar{K}\cdot\bar{r}}+R^{TE}\,\hat{e}(k_{0_z})\,e^{k_{0_z}}\,e^{i\bar{k}\cdot\bar{r}} \tag{30a}$$
$$\bar{K}_h = \hat{h}(-k_{0_z})\,e^{i\bar{K}\cdot\bar{r}}+R^{TM}\,\hat{h}(k_{0_z})\,e^{k_{0_z}}\,e^{i\bar{k}\cdot\bar{r}} \tag{30b}$$

2. For region ℓ, $i=\ell$:

$$\bar{K}_e = A_\ell\,\hat{e}(k_{\ell_z})\,e^{i\bar{k}_\ell\cdot\bar{r}}+B_\ell\,\hat{e}(-k_{\ell_z})\,e^{i\bar{K}_\ell\cdot\bar{r}} \tag{31a}$$
$$\bar{K}_h = C_\ell\,\hat{h}(k_{\ell_z})\,e^{i\bar{k}_\ell\cdot\bar{r}}+D_\ell\,\hat{h}(-k_{\ell_z})\,e^{i\bar{K}_\ell\cdot\bar{r}} \tag{31b}$$
$$\tag{31c}$$

3. For region t, $i=t$:

$$\bar{K}_e = T^{TE}\,\hat{e}(-k_{t_z})\,e^{i\bar{K}_t\cdot\bar{r}} \tag{32a}$$
$$\bar{K}_h = T^{TM}\,\hat{h}(-k_{t_z})\,e^{i\bar{K}_t\cdot\bar{r}} \tag{32b}$$
$$\tag{32c}$$

where

$$k_{\ell_z} = \sqrt{k_\ell^2-k_x^2-k_y^2}\,, \tag{33a}$$
$$\bar{k}_\ell = \hat{x}k_x+\hat{y}k_y+\hat{z}k_{\ell_z}\,, \tag{33b}$$
$$\bar{K}_\ell = \hat{x}k_x+\hat{y}k_y-\hat{z}k_{\ell_z}\,, \tag{33c}$$

and the coefficients A_ℓ, B_ℓ, C_ℓ and D_ℓ are determined from the boundary conditions.

The boundary conditions apply to the tangential electric and magnetic fields. Thus, in terms of Green's functions, we need to satisfy the continuity of $\hat{z}\times\overline{\overline{G}}(\bar{r},\bar{r}')$ and $\hat{z}\times\nabla\times$

$\overline{\overline{G}}(\bar{r},\bar{r}')$. Let us write this at the interface between media (ℓ) and ($\ell+1$), by separating the TE and TM components:

$$A_\ell\, e^{ik_{\ell_z} z} + B_\ell\, e^{-ik_{\ell_z} z} = A_{\ell+1}\, e^{ik_{\ell_z} z} + B_{\ell+1}\, e^{-ik_{\ell_z} z} \tag{34a}$$

$$k_{\ell_z}\left[A_\ell\, e^{ik_{\ell_z} z} - B_\ell\, e^{-ik_{\ell_z} z}\right] = k_{z\ell+1}\left[A_{\ell+1}\, e^{ik_{\ell_z} z} - B_{\ell+1}\, e^{-ik_{\ell_z} z}\right] \tag{34b}$$

$$\frac{k_{\ell_z}}{k_\ell}\left[A_\ell\, e^{ik_{\ell_z} z} - B_\ell\, e^{-ik_{\ell_z} z}\right] = \frac{k_{z\ell+1}}{k_{\ell+1}}\left[A_{\ell+1}\, e^{ik_{\ell_z} z} - B_{\ell+1}\, e^{-ik_{\ell_z} z}\right] \tag{34c}$$

$$k_{\ell_z}\left[C_\ell\, e^{ik_{\ell_z} z} + D_\ell\, e^{-ik_{\ell_z} z}\right] = k_{z\ell+1}\left[C_{\ell+1}\, e^{ik_{\ell_z} z} + D_{\ell+1}\, e^{-ik_{\ell_z} z}\right] \tag{34d}$$

With the conditions in the first and last layer as:

$$A_0 = R^{TE}\,, \quad B_0 = 1\,, \qquad\qquad C_0 = R^{TM}\,, \quad D_0 = 1\,, \tag{35a}$$

$$A_t = 0\,, \quad B_t = T^{TE}\,, \qquad\qquad C_t = 0\,, \quad D_t = T^{TM}\,. \tag{35b}$$

Before evaluating these coefficients, we can build a recursive scheme to calculate the amplitudes from region ℓ to region $\ell+1$.

For example, it is straightforward to build a propagation matrix for TE modes from Eq. (34a) and (34b):

$$\begin{pmatrix} A_{\ell+1}\, e^{ik_{z\ell+1}d_{\ell+1}} \\ B_{\ell+1}\, e^{-ik_{z\ell+1}d_{\ell+1}} \end{pmatrix} = \overline{\overline{V}}^{TE} \begin{pmatrix} A_\ell\, e^{ik_{z\ell}d_\ell} \\ B_\ell\, e^{-ik_{z\ell}d_\ell} \end{pmatrix} \tag{36}$$

A similar procedure of course applied to the TM modes:

$$\begin{pmatrix} C_{\ell+1}\, e^{ik_{z\ell+1}d_{\ell+1}} \\ D_{\ell+1}\, e^{-ik_{z\ell+1}d_{\ell+1}} \end{pmatrix} = \overline{\overline{V}}^{TM} \begin{pmatrix} C_\ell\, e^{ik_{z\ell}d_\ell} \\ D_\ell\, e^{-ik_{z\ell}d_\ell} \end{pmatrix} \tag{37}$$

In order to end up the recursive method, we have to express the reflection and transmission coefficient in the first and last regions, respectively. We shall only illustrated this point here, as it has been developed in previous classes.

Let us consider a plane wave incident from region 0, with its plane of incidence parallel to the (xy) plane. All fields vectors are independent on y, so that $\frac{\partial}{\partial y} = 0$ in Maxwell's equations. Thus, we can decompose the fields into their TE and TM components. We get in region ℓ:

- TE modes:

$$H_{\ell x} = -\frac{1}{i\omega\mu_\ell}\frac{\partial}{\partial z}E_{\ell y}\,, \tag{38a}$$

$$H_{\ell z} = \frac{1}{i\omega\mu_\ell}\frac{\partial}{\partial x}E_{\ell y}\,, \tag{38b}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} + \omega^2\epsilon_\ell\mu_\ell\right)E_{\ell y} = 0\,. \tag{38c}$$

- TM modes:

$$
\begin{aligned}
E_{\ell x} =& \frac{1}{i\omega\epsilon_\ell}\frac{\partial}{\partial z}H_{\ell y}\,, &(39a)\\
E_{\ell z} =& -\frac{1}{i\omega\epsilon_\ell}\frac{\partial}{\partial x}H_{\ell y}\,, &(39b)\\
\left(\frac{\partial^2}{\partial x^2}+\right.&\left.\frac{\partial}{\partial y}+\omega^2\epsilon_\ell\mu_\ell\right)H_{\ell y}=0\,. &(39c)
\end{aligned}
$$

For a TE wave inside the stratified medium:

$$
\begin{aligned}
E_{\ell y} =& (A_\ell e^{ik_{\ell_z}z} + B_\ell e^{-ik_{\ell_z}z})\, e^{ik_x x}\,, &(40a)\\
H_{\ell x} =& -\frac{k_{\ell_z}}{\omega\mu_\ell}(A_\ell e^{ik_{\ell_z}z} - B_\ell e^{-ik_{\ell_z}z})\, e^{ik_x x}\,, &(40b)\\
H_{\ell z} =& \frac{k_x}{\omega\mu_\ell}(A_\ell e^{ik_{\ell_z}z} + B_\ell e^{-ik_{\ell_z}z})\, e^{ik_x x}\,. &(40c)
\end{aligned}
$$

By matching the boundary conditions, and upon using the already known notation

$$
\begin{aligned}
p_{\ell(\ell+1)} =& \frac{\mu_\ell k_{z(\ell+1)}}{\mu_{\ell+1}k_{\ell_z}}\,, &(41a)\\
R_{\ell(\ell+1)} =& \frac{1-p_{\ell(\ell+1)}}{1+p_{\ell(\ell+1)}}\,. &(41b)
\end{aligned}
$$

we get the recursive relation:

$$
\frac{A_\ell}{B_\ell} = \frac{e^{2ik_{\ell_z}d_\ell}}{R_{\ell(\ell+1)}} + \frac{[1-1/R^2_{\ell(\ell+1)}]\, e^{2i[k_{(\ell+1)_z}+k_{\ell_z}]d_\ell}}{1/R_{\ell(\ell+1)}\, e^{2ik_{(\ell+1)_z}d_\ell} + \frac{A_{\ell+1}}{B_{\ell+1}}}\,, \tag{42}
$$

with the limiting condition:

$$
\frac{A_t}{B_t} = 0,, \qquad \frac{A_0}{B_0} = R\,. \tag{43}
$$

Example: for a two-layer medium ($t = 2$):

$$
R = \frac{R_{01} + R_{12}e^{2ik_{1_z}(d_1-d_0)}}{1 + R_{01}R_{12}e^{2ik_{1_z}(d_1-d_0)}}e^{2ik_z d_0}\,. \tag{44}
$$