Integral Equations in Electromagnetics

Massachusetts Institute of Technology
6.635 lecture notes

Most integral equations do not have a closed form solution. However, they can often be discretized and solved on a digital computer.

Proof of the existence of the solution to an integral equation by discretization was first presented by Fredholm in 1903.

In general, integral equations can be divided into two families:

1. When the unknown is in the integral only, the integral equation is called of *the first kind*.
2. When the unknown is both inside and outside the integral, the integral equation is called of *the second kind*.

For electromagnetic applications, we can have both scalar and vector integral equations.

1 Scalar integral equations

Let us consider the situation of Fig. 1: two regions are defined in space, region 2 bounded by the closed surface S and region 1 being all the remaining space, bounded by S and S_∞, in which sources are located.

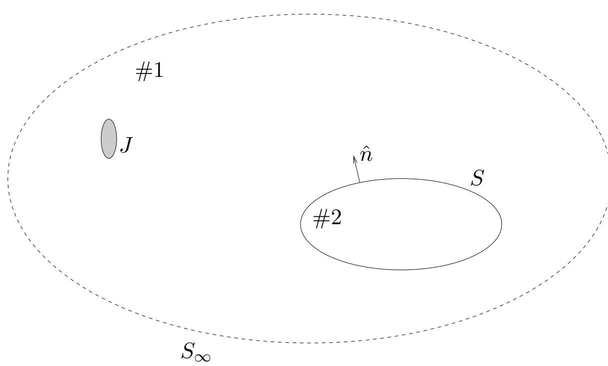

Figure 1: Definition of the geometry of the problem.

It is already known that we can write:

$$(\nabla^2 + k_1^2)\,\Phi_1(\bar{r}) = J(\bar{r})\,, \tag{1a}$$
$$(\nabla^2 + k_2^2)\,\Phi_2(\bar{r}) = 0\,, \tag{1b}$$

and similarly for the Green's functions

$$(\nabla^2 + k_1^2)\, g_1(\bar r, \bar r') = -\delta(\bar r - \bar r') \qquad \text{in region 1,} \tag{2a}$$

$$(\nabla^2 + k_2^2)\, g_2(\bar r, \bar r') = -\delta(\bar r - \bar r') \qquad \text{in region 2.} \tag{2b}$$

Upon performing (Eq. (1a)$g_1(\bar r, \bar r')$−Eq. (2b)$\Phi_1(\bar r, \bar r')$), we get:

$$[g_1(\bar r, \bar r')\, \nabla^2 \Phi_1(\bar r) - \Phi_1(\bar r)\, \nabla^2 g_1(\bar r, \bar r')] = J(\bar r) g_1(\bar r, \bar r') + \delta(\bar r - \bar r')\Phi_1(\bar r)\,. \tag{3}$$

Upon integrating Eq. (3) over the entire volume and using the identity $\nabla \cdot (g\nabla\Phi - \Phi\nabla g) = g\nabla^2\Phi - \Phi\nabla^2 g$, we get:

$$\int_V \mathrm{d}v\ \nabla \cdot [g_1(\bar r, \bar r')\Phi_1(\bar r) - \Phi_1(\bar r)\nabla g_1(\bar r, \bar r')] = \int_V \mathrm{d}v J(\bar r)\, g(\bar r, \bar r') + \Phi_1(\bar r')\,. \tag{4}$$

By Gauss theorem, we reduce the left-hand side integral to a surface integral. Also

$$\int_V \mathrm{d}v J(\bar r) g_1(\bar r, \bar r') = -\Phi_{\mathrm{inc}}(\bar r')\,. \tag{5}$$

We therefore obtain

$$-\int_{S+S_\infty} \mathrm{d}s\, \hat n \cdot [g_1(\bar r, \bar r')\nabla\Phi_1(\bar r) - \Phi_1(\bar r)\nabla g_1(\bar r, \bar r')] = -\Phi_{\mathrm{inc}}(\bar r) + \Phi_1(\bar r')\,, \qquad \bar r' \in V\,. \tag{6}$$

By invoking the radiation condition, the integral over S_∞ vanishes, leaving (and exchanging $\bar r$ and $\bar r'$ so that primed coordinates correspond to sources and unprimed ones to observation):

$$\Phi_1(\bar r) = \Phi_{\mathrm{inc}}(\bar r) - \int_S \mathrm{d}s'\, \hat n \cdot [g_1(\bar r, \bar r')\nabla'\Phi_1(\bar r') - \Phi_1(\bar r')\nabla' g_1(\bar r, \bar r')] \qquad \bar r \in V_1\,. \tag{7}$$

For $\bar r \in V_2$, the wave equation has no source and therefore the integration of the delta function yields a zero value. Performing the same steps for this second case, we get the generic relation:

$$\Phi_{\mathrm{inc}}(\bar r) - \int_S \mathrm{d}s'\, \hat n \cdot [g_1(\bar r, \bar r')\nabla'\Phi_1(\bar r') - \Phi_1(\bar r')\nabla' g_1(\bar r, \bar r')] = \begin{cases} \Phi_1(\bar r) & \bar r \in V_1 \\ 0 & \bar r \in V_2 \end{cases} \tag{8}$$

This is directly evocative of Huygens' principle:

- For $\bar r \in V_1$: the total field $\Phi_1(\bar r)$ is the sum of the incident field plus the field due to the surface currents on the surface S.
- For $\bar r \in V_2$: the surface source on S produces a field that exactly opposes Φ_{inc}, yielding the extinction theorem.

Applying the same reasoning to region 2, we write (where there is no incident field):

$$\int_S \mathrm{d}s'\, \hat n \cdot [g_2(\bar r, \bar r')\nabla'\Phi_2(\bar r') - \Phi_2(\bar r')\nabla' g_2(\bar r, \bar r')] = \begin{cases} 0 & \bar r \in V_1 \\ \Phi_2(\bar r) & \bar r \in V_2 \end{cases} \tag{9}$$

Note that the sign reversal is due to the definition of the normal vector $\hat{n}$ which has to point outward from the surface. Here, since we use the same $\hat{n}$ as before, we have to take it as being negative.

Eqs. (8) and (9) have four independent unknowns:

$$\Phi_1,\ \Phi_2\,;\hat{n}\cdot\nabla\Phi_1,\ \hat{n}\cdot\nabla\Phi_2\,, \tag{10}$$

which can be related by the boundary conditions. Here, $g(\bar{r},\bar{r}')$ and $\hat{n}\cdot\nabla g(\bar{r},\bar{r}')$ are the kernel of the integral equation.

2 Vector integral equation

For the sake of completeness, we shall write the vector wave equation as well, although we will not use it here directly.

Considering the same situation as above, we know that the fields have to satisfy:

$$\nabla\times\nabla\times\bar{E}_1(\bar{r})-\omega^2\epsilon_1\mu_1\bar{E}_1(\bar{r}) = i\omega\mu_1\bar{J}(\bar{r})\,, \tag{11a}$$

$$\nabla\times\nabla\times\bar{E}_2(\bar{r})-\omega^2\epsilon_2\mu_2\bar{E}_2(\bar{r}) = 0\,, \tag{11b}$$

and the Green's functions:

$$\nabla\times\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}')-\omega^2\epsilon_1\mu_1\overline{\overline{G}}_1(\bar{r},\bar{r}') = \overline{\overline{I}}\delta(\bar{r}-\bar{r}')\,, \tag{12a}$$

$$\nabla\times\nabla\times\overline{\overline{G}}_2(\bar{r},\bar{r}')-\omega^2\epsilon_2\mu_2\overline{\overline{G}}_2(\bar{r},\bar{r}') = \overline{\overline{I}}\delta(\bar{r}-\bar{r}')\,. \tag{12b}$$

By the same technique as before ($\int_V \mathrm{d}v\ [Eq.\ (11a)\cdot\overline{\overline{G}}_1(\bar{r},\bar{r}') - Eq.\ (12a)\cdot\bar{E}_1(\bar{r})]\mathrm{d}v$)

$$\int_V \mathrm{d}v\left[\nabla\times\nabla\times\bar{E}_1(\bar{r})\cdot\overline{\overline{G}}_1(\bar{r},\bar{r}')-\bar{E}_1(\bar{r})\cdot\nabla\times\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}')\right] = i\omega\mu_1\int_V \mathrm{d}v\ \bar{J}(\bar{r})\cdot\overline{\overline{G}}_1(\bar{r},\bar{r}')-\bar{E}_1(\bar{r})\,. \tag{13}$$

The left-hand side can be transformed into a surface integral (left as exercise) and the right-hand side written in terms of incident field, yielding:

$$\bar{E}_1(\bar{r}') = \bar{E}_{\mathrm{inc}}(\bar{r}')+\int_{S+S_\infty}\mathrm{d}s\ \hat{n}\cdot\left\{[\nabla\times\bar{E}_1(\bar{r})]\times\overline{\overline{G}}_1(\bar{r},\bar{r}')+\bar{E}_1(\bar{r})\times\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}')\right\},\quad \bar{r}'\in V_1\,. \tag{14}$$

By reciprocity of the Green's functions, we can transform

$$\begin{aligned}\hat{n}\cdot[\nabla\times\bar{E}_1(\bar{r})]\times\overline{\overline{G}}_1(\bar{r},\bar{r}') &= \hat{n}\times[\nabla\times\bar{E}_1(\bar{r})]\cdot\overline{\overline{G}}_1(\bar{r},\bar{r}')\\ &= i\omega\mu_1\overline{\overline{G}}_1(\bar{r},\bar{r}')\cdot\hat{n}\times\bar{H}_1(\bar{r})\,.\end{aligned} \tag{15}$$

In addition, for an unbounded homogeneous medium ($\nabla\times\overline{\overline{G}}(\bar{r},\bar{r}')$ is reciprocal):

$$\begin{aligned}\hat{n}\cdot\bar{E}_1(\bar{r})\times\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}') &= \hat{n}\times\bar{E}_1(\bar{r})\cdot\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}')\\ &= -[\nabla\times\overline{\overline{G}}_1(\bar{r},\bar{r}')]\cdot\hat{n}\times\bar{E}_1(\bar{r})\,.\end{aligned} \tag{16}$$

For Green's functions that satisfy the radiation condition, Eq. (14) becomes:

$$\bar{E}_1(\bar{r}') = \bar{E}_{\text{inc}}(\bar{r}') + \int_S \mathrm{d}s\, \hat{n} \cdot \left\{ i\omega\mu_1 \overline{\overline{G}}_1(\bar{r},\bar{r}') \cdot \hat{n} \times \bar{H}_1(\bar{r}) - [\nabla \times \overline{\overline{G}}_1(\bar{r},\bar{r}')] \cdot \hat{n} \times \bar{E}_1(\bar{r}) \right\}. \tag{17}$$

We perform the same steps for the other region to eventually obtain (and again interchanging $\bar{r}$ and $\bar{r}'$):

$$\bar{E}_{\text{inc}}(\bar{r}) + \int_S \mathrm{d}s' \left\{ i\omega\mu_1 \overline{\overline{G}}_1(\bar{r},\bar{r}') \cdot \hat{n} \times \bar{H}_1(\bar{r}') - [\nabla' \times \overline{\overline{G}}_1(\bar{r},\bar{r}')] \cdot \hat{n} \times \bar{E}_1(\bar{r}') \right\} = \begin{cases} \bar{E}_1(\bar{r}) & \bar{r} \in V_1 \\ 0 & \bar{r} \in V_2 \end{cases} \tag{18a}$$

$$-\int_S \mathrm{d}s' \left\{ i\omega\mu_2 \overline{\overline{G}}_2(\bar{r},\bar{r}') \cdot \hat{n} \times \bar{H}_2(\bar{r}') - [\nabla' \times \overline{\overline{G}}_2(\bar{r},\bar{r}')] \cdot \hat{n} \times \bar{E}_2(\bar{r}') \right\} = \begin{cases} 0 & \bar{r} \in V_1 \\ \bar{E}_2(\bar{r}) & \bar{r} \in V_2 \end{cases} \tag{18b}$$

Together with the boundary conditions

$$\hat{n} \times \bar{H}_1(\bar{r}) = \hat{n} \times \bar{H}_2(\bar{r})\,, \tag{19a}$$

$$\hat{n} \times \bar{E}_1(\bar{r}) = \hat{n} \times \bar{E}_2(\bar{r})\,, \tag{19b}$$

this system can be solved for $\hat{n} \times \bar{E}_1(\bar{r})$ and $\hat{n} \times \bar{H}_1(\bar{r})$.

Note also that Eqs. (18) can be written in terms of electric and magnetic currents, and magnetic Green's functions:

$$\bar{J}_{eq}(\bar{r}') = \hat{n} \times \bar{H}_1(\bar{r}')\,; \qquad -\bar{M}_{eq}(\bar{r}') = \hat{n} \times \bar{E}_1(\bar{r}')\,, \tag{20a}$$

so that for example:

$$\bar{E}_{\text{inc}}(\bar{r}) + \int_S \mathrm{d}s' \left\{ i\omega\mu_1 \overline{\overline{G}}_{e_1}(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}') + \overline{\overline{G}}_{m_1}(\bar{r},\bar{r}') \cdot \bar{M}(\bar{r}') \right\} = \begin{cases} \bar{E}_1(\bar{r}) & \bar{r} \in V_1 \\ 0 & \bar{r} \in V_2 \end{cases} \tag{21}$$

3 Problem with the internal resonance

A question arises: is the integral equation equivalent to Maxwell's equations? Or asked differently, if we solve the integral equation and Maxwell's equations, do we get the same solution?

The answer is actually "*no*", that is they are not always equivalent to each-other. The problem comes from spurious solutions at frequencies corresponding to the eigenfrequencies of the cavity enclosed by the surface S. This problem is generally referred to as the "internal resonance of the integral equation".

However, this lack of complete equivalence between the physical problem and its defining integral equation is rather minor and infrequent phenomenon, and is therefore often tolerated in practice.

4 Scattering by a rough surface

Let us consider the 2D problem (for which we shall use a scalar integral equation) depicted in Fig. 2.

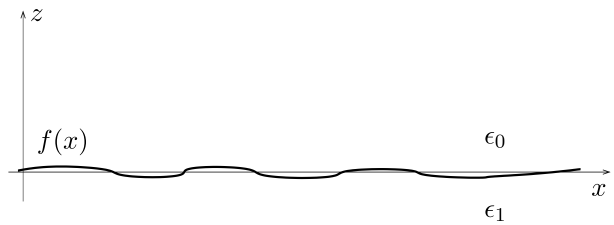

Figure 2: Rough surface S separating two media.

The integral equation in scalar form is given by:

$$\Phi_{\text{inc}}(\bar{r}) + \int_S \mathrm{d}s' \hat{n} \cdot [\Phi(\bar{r}')\nabla g(\bar{r},\bar{r}') - g(\bar{r},\bar{r}')\nabla\Phi(\bar{r}')] = \begin{cases} \Phi(\bar{r}) & \bar{r} \in V_0 \\ 0 & \bar{r} \in V_1 \end{cases} \tag{22}$$

4.1 Dirichlet boundary conditions: EFIE

For a TE wave ($\bar{E} = E\hat{y}$) and PEC surface, the boundary condition is

$$\Phi(\bar{r}) = 0\,, \qquad \text{for } \bar{r} \in S\,. \tag{23}$$

The integral equation becomes, for $\bar{r} \in S\,,\ \bar{r}' \in S$:

$$\Phi_{\text{inc}}(\bar{r}) - \int_S \mathrm{d}s'\ g(\bar{r},\bar{r}')\ \hat{n} \cdot \nabla\Phi(\bar{r}')\ = \begin{cases} \Phi(\bar{r}) \\ 0 \end{cases} \tag{24}$$

This equation, in which Φ represent the electric field, is referred to as the *electric field integral equation* (EFIE).

Note that as $\bar{r}$ gets closer to the surface, $\Phi(\bar{r}) \to 0$ (from the boundary condition) so that we do not need to distinguish between approaching the surface from one side or the other. In fact, we can unify the equations and write:

$$\Phi_{\text{inc}}(\bar{r}) - \int_S \mathrm{d}s'\ g(\bar{r},\bar{r}')\ \hat{n} \cdot \nabla\Phi(\bar{r}')\ = 0\,, \qquad \bar{r} \in S\,, \bar{r}' \in S\,. \tag{25}$$

In addition, $g(\bar{r},\bar{r}')$ has an integrable singularity as $\bar{r} \to \bar{r}'$. Let us consider the surface depicted in Fig. 2, with $z = f(x)$:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}z^2} = \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\,, \tag{26}$$

such that the integral equation becomes:

$$\Phi_{\text{inc}}(\bar{r}) = \int_{\Delta x} \mathrm{d}x'\sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\, g(x, f(x), x', f(x'))\,(\hat{n} \cdot \nabla\Phi(\bar{r}'))\,, \qquad \text{at } z' = f(x')\,, \tag{27}$$

where we can limit Δx to $[-L/2, L/2]$.

By letting

$$\sqrt{1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\,(\hat{n}\cdot\nabla\Phi(\bar{r}'))|_{z'=f(x')} = U(x')\,, \tag{28a}$$

$$\Phi_{\mathrm{inc}}(x, f(x)) = b(x)\,, \tag{28b}$$

$$K(x,x') = g(x, f(x), x', f(x'))\,, \tag{28c}$$

we can rewrite the integral equation as

$$\int_{-L/2}^{L/2} \mathrm{d}x'\, K(x,x')\, U(x') = b(x)\,, \tag{29}$$

which has to be solved numerically.

Before doing this, we shall write the integral equation with Neumann boundary conditions.

4.2 Neumann boundary conditions: MFIE

This corresponds to a TM case with $\bar{H} = H\hat{y}$ with a PEC surface. In this case, the boundary condition is

$$\hat{n}\cdot\nabla\Phi(\bar{r}) = 0\,, \tag{30}$$

and the integral equation becomes

$$\Phi_{\mathrm{inc}}(\bar{r}) + \int_S \mathrm{d}s'\;\Phi(\bar{r})\,\hat{n}\cdot\nabla g(\bar{r},\bar{r}') \;=\; \begin{cases}\Phi(\bar{r}) & \bar{r}\in V_0\\ 0 & \bar{r}\in V_1\end{cases} \tag{31}$$

In this case, it makes a difference if we approach the surface from the top or from the bottom. In fact

$$\Phi_{\mathrm{inc}}(\bar{r}_+) + \int_S \mathrm{d}s'\Phi(\bar{r}')\;\hat{n}\cdot\nabla g(\bar{r}_+,\bar{r}') = \Phi(\bar{r}_+)\,, \tag{32a}$$

$$\Phi_{\mathrm{inc}}(\bar{r}_-) + \int_S \mathrm{d}s'\Phi(\bar{r}')\;\hat{n}\cdot\nabla g(\bar{r}_-,\bar{r}') = 0\,. \tag{32b}$$

These two equations seems inconsistent with one another. We shall show that in fact, they are actually consistent with each-other, due to the singularity of the Green's functions.

We shall examine what happens when we let $\bar{r}$ approach the surface. Fig. 3 is an illustration of the situation at the immediate vicinity of the surface. The integral part of the equation can be written as:

$$\int_S \mathrm{d}s'\Phi(\bar{r}')\,\hat{n}\cdot\nabla g(\bar{r},\bar{r}') = PV\int_S \mathrm{d}s'\Phi(\bar{r}')\,\hat{n}\cdot\nabla g(\bar{r},\bar{r}') + \int_{\mathrm{piece}} \mathrm{d}s'\Phi(\bar{r}')\,\hat{n}\cdot\nabla g(\bar{r},\bar{r}')\,, \tag{33}$$

where PV denotes the principal value and 'piece' refers to the integration over the local domain shown in Fig. 3. For this integral, we use the local coordinates:

$$\mathrm{d}s' = \mathrm{d}X'\,, \qquad \hat{n} = \hat{Z}'\,,\mathrm{etc} \tag{34}$$

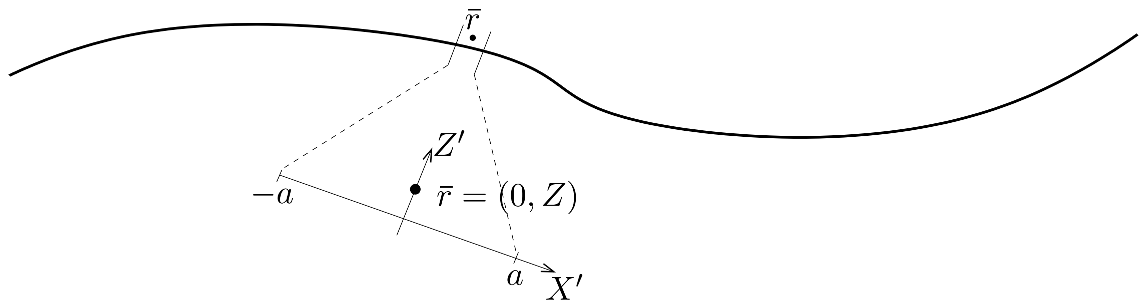

Figure 3: Zoom on the rough surface.

The integral becomes:

$$\int_{\text{piece}} \mathrm{d}s' \Phi(\bar{r}') \, \hat{n} \cdot \nabla g(\bar{r}, \bar{r}') = \lim_{a \to 0} \lim_{|Z| \to 0} \Phi(\bar{r}) \int_{-a}^{a} \mathrm{d}X' \frac{\partial}{\partial Z'} g(\bar{r}, \bar{r}') \,. \tag{35}$$

Over the small piece, we have:

$$|\bar{r} - \bar{r}'| = \sqrt{X'^2 + (Z - Z')^2} \,, \tag{36a}$$

$$g = \frac{i}{4} H_0^{(1)}(|\bar{r} - \bar{r}'|) \;\to\; -\frac{1}{2\pi} \ln(\frac{|\bar{r} - \bar{r}'|}{2}) \,, \tag{36b}$$

$$\left(\frac{\partial g}{\partial Z'} \right)_{Z'=0} = \frac{Z}{2\pi (X'^2 + Z^2)} \,. \tag{36c}$$

Thus, the integral becomes

$$\begin{aligned} \int_{\text{piece}} &= \lim_{a \to 0} \lim_{|Z| \to 0} \Phi(\bar{r}) \int_{-a}^{a} \mathrm{d}X' \frac{1}{2\pi} \frac{Z}{X'^2 + Z^2} \\ &= \lim_{a \to 0} \lim_{|Z| \to 0} \frac{\Phi(\bar{r})}{2\pi} \left[\tan^{-1} \frac{X'}{Z} \right]_{-a}^{a} \\ &= \begin{cases} \frac{1}{2} \Phi(\bar{r}) & \text{for } Z > 0 \\ -\frac{1}{2} \Phi(\bar{r}) & \text{for } Z < 0 \end{cases} \end{aligned} \tag{37}$$

The two parts of the integral then become:

$$\Phi_{\text{inc}}(\bar{r}) + PV \int_S \mathrm{d}s' \Phi(\bar{r}') \, \hat{n} \cdot \nabla g(\bar{r}, \bar{r}') + \frac{1}{2} \Phi(\bar{r}) = \Phi(\bar{r}) \,, \tag{38a}$$

$$\Phi_{\text{inc}}(\bar{r}) + PV \int_S \mathrm{d}s' \Phi(\bar{r}') \, \hat{n} \cdot \nabla g(\bar{r}, \bar{r}') - \frac{1}{2} \Phi(\bar{r}) = 0 \,, \tag{38b}$$

$$\tag{38c}$$

which is cast into one equation:

$$\Phi_{\text{inc}}(\bar{r}) + PV \int_S \mathrm{d}s' \Phi(\bar{r}') \, \hat{n} \cdot \nabla g(\bar{r}, \bar{r}') = \frac{1}{2} \Phi(\bar{r}) \,. \tag{39}$$

Eq. (39) is called the *magnetic field integral equation* (since Φ represents the magnetic field) and is an integral equation of the second kind.

In the same way as before, we can define the kernel as

$$K(x,x') = \sqrt{1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\,\hat{n}\cdot\nabla g(\bar{r},\bar{r}')|_{z=f(x),z'=f(x')}\,, \tag{40}$$

and write the MFIE as

$$\Phi_{\rm inc}(\bar{r}) + PV\int_{-\infty}^{\infty}\mathrm{d}x'\,\Phi(x')\,K(x,x') = \frac{1}{2}\,\Phi(x)\,. \tag{41}$$

5 Solving the EFIE

Upon using the notation introduced in Section 4.1, the problem is reduced to solving the following integral equation:

$$\int_{-L/2}^{L/2}\mathrm{d}x'\,K(x,x')\,U(x') = b(x)\,. \tag{42}$$

Let us subdivide the integration domain into N small elements, each of length $\Delta = L/N$, and centered at x_m ($m\in[1,N]$). Thus, constraining the observation at these discrete locations, the integral equation becomes

$$\int_{-L/2}^{L/2}\mathrm{d}x'\,K(x_m,x')\,U(x') = b(x_m)\,. \tag{43}$$

Next, if we suppose that $U(x)$ is constant in each interval, we replace the integral by a sum over all segments, excluding the singular term:

$$\Delta x\sum_{\substack{n=1\\ n\neq m}} K(x_m,x_n)\,U(x_n) + U(x_m)\int_m \mathrm{d}x' K(x_m,x') = b(x_m)\,. \tag{44}$$

Note that we have to single out the singularity, *i.e.* the m^{th} interval because $K(x_m,x')$ is singular at $x'=x_m$. This part is known as the self-patch contribution.

For a 2D problem, we have:

$$K(x_m,x') = \frac{i}{4}\,H_0^{(1)}(k\sqrt{(x_m-x')^2+(f(x_m)-f(x'))^2})\,. \tag{45}$$

Upon approximating $f(x')-f(x_m)\simeq f'(x_m)(x'-x_m)$, we write:

$$K(x_m,x') = \frac{i}{4}\,H_0^{(1)}(k|x'-x_m|\sqrt{1+f'(x_m)^2})\,. \tag{46}$$

For small argument: $H_0^{(1)}(\alpha)\approx 1+i\frac{2}{\pi}\ln(\alpha\gamma/2)$ where γ is the Euler constant ($\gamma\simeq 1.78$). Therefore:

$$K(x_m,x') = \frac{i}{4}\left[1+i\frac{1}{\pi}\ln\left(\frac{\gamma}{2}k(x'-x_m)\sqrt{1+f'^2(x_m)}\right)\right]. \tag{47}$$

Therefore, the integral becomes:

$$\begin{aligned}\int_m \mathrm{d}x' K(x_m, x') &= 2\int_{x_m}^{x_m+\Delta x/2} \mathrm{d}x' K(x_m, x') = 2\int_0^{\Delta x/2} \mathrm{d}x' K(x_m, x' + x_m) \\ &\simeq \frac{i}{2}\int_0^{\Delta x/2} \mathrm{d}x' \left[1 + i\frac{1}{\pi}\ln\left(\frac{\gamma}{2}kx'\sqrt{1+f'^2(x_m)}\right)\right] \\ &= \frac{i\Delta x}{4}\left\{1 + i\frac{2}{\pi}\ \ln\left(\frac{\gamma k}{4e}\Delta x\sqrt{1+f'^2(x_m)}\right)\right\}, \end{aligned} \tag{48}$$

where $\ln(e) = 1$.

We can therefore cast the integral equation into a matrix equation of the form:

$$\sum_{n=1}^{N} A_{mn}U_n = b_m\,, \tag{49}$$

where

$$U_n = U(x_n) \text{ is the unknown,} \tag{50a}$$

$$b_m = b(x_m)\,, \tag{50b}$$

$$A_{mn} = \begin{cases} \Delta x\ K(x_m, x_n) & \text{for } m \neq n \\ \frac{i\Delta x}{4}\left\{1 + i\frac{2}{\pi}\ \ln\left(\frac{\gamma k}{4e}\Delta x\sqrt{1+f'^2(x_m)}\right)\right\} & \text{for } m = n \end{cases} \tag{50c}$$

The system can now easily be solved numerically.