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6.641 Electromagnetic Fields, Forces, and Motion  
Spring 2009

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**Lecture 19: Waves and Instabilities in the Presence of Motion**

I. Galilean Transformation

$$\xi = \xi'$$

Waves and Instabilities in the Presence of Material Motion

The fixed coordinate system  $x, y, z$  or the moving coordinate system  $x', y', z'$  can be used to define a given position along the equilibrium axis of the string. Because the moving coordinate system has the same velocity  $U$  in the  $x$ -direction as the string, the string appears to move only in the  $z'$ -direction when viewed from the primed frame.

$$\begin{aligned} x &= x' + Ut' \\ z &= z' \\ t &= t' \end{aligned} \Rightarrow \frac{\partial \xi(x, t)}{\partial t'} = \frac{\partial \xi}{\partial t} \frac{\partial t}{\partial t'} + \frac{\partial \xi}{\partial x} \frac{\partial x}{\partial t'}$$

$$\frac{\partial \xi}{\partial t'} = \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}$$

$$\frac{\partial^2 \xi}{\partial t'^2} = \frac{\partial}{\partial t} \left[ \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right] + U \frac{\partial}{\partial x} \left[ \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right] = \frac{\partial^2 \xi}{\partial t^2} + 2U \frac{\partial^2 \xi}{\partial x \partial t} + U^2 \frac{\partial^2 \xi}{\partial x^2}$$

$$= \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi$$

$$\frac{\partial \xi(\mathbf{x}, t)}{\partial \mathbf{x}'} = \frac{\partial \xi}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{x}'_1} + \frac{\partial \xi}{\partial t} \frac{\partial t}{\partial \mathbf{x}'_0}$$

$$\frac{\partial \xi}{\partial \mathbf{x}'} = \frac{\partial \xi}{\partial \mathbf{x}} \Rightarrow \frac{\partial^2 \xi}{\partial \mathbf{x}'^2} = \frac{\partial^2 \xi}{\partial \mathbf{x}^2}$$

$$m \frac{\partial^2 \xi'}{\partial t'^2} = T \frac{\partial^2 \xi'}{\partial \mathbf{x}'^2} + F'_{\text{ext}}$$

$$m \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial \mathbf{x}} \right)^2 \xi = T \frac{\partial^2 \xi}{\partial \mathbf{x}^2} + F_{\text{ext}}$$

## II. Fast and Slow Non Dispersive waves ( $F_{\text{ext}} = 0$ )

$$\xi(\mathbf{x}, t) = \text{Re} \left[ \hat{\xi} e^{j(\omega t - kx)} \right]$$

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial \mathbf{x}} \right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial \mathbf{x}^2} ; \quad v_s^2 = T/m$$

$$+(\omega - kU)^2 = +k^2 v_s^2$$

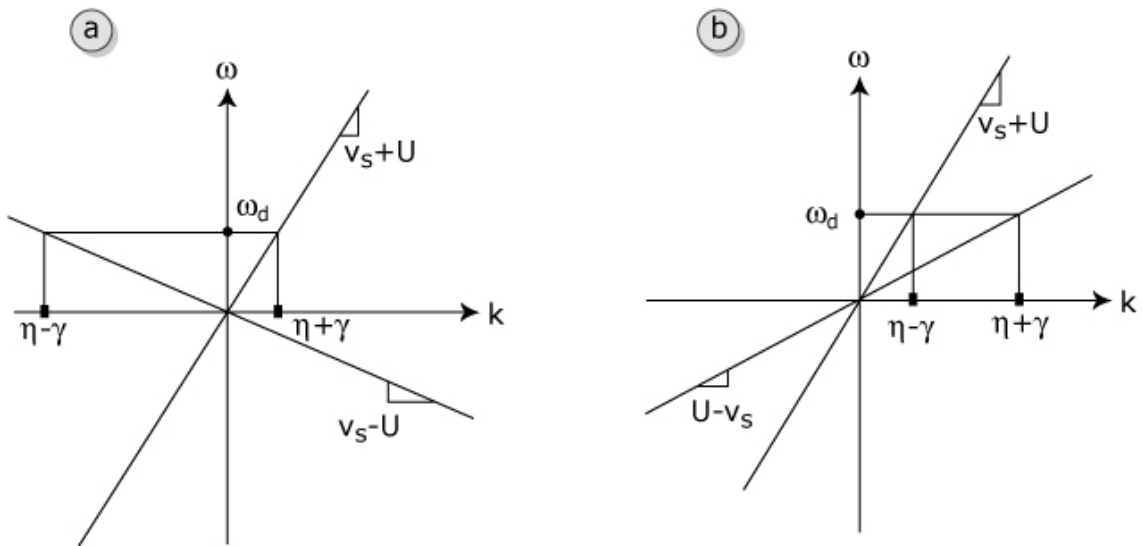
$$\omega = k(U \pm v_s)$$

$$k = \frac{\omega}{U \pm v_s} \frac{(U \mp v_s)}{(U \mp v_s)}$$

$$= \frac{\omega(U \mp v_s)}{U^2 - v_s^2}$$

$$= \eta \pm \gamma$$

$$\eta = \frac{\omega U}{U^2 - v_s^2}, \quad \gamma = \frac{\omega v_s}{U^2 - v_s^2}$$



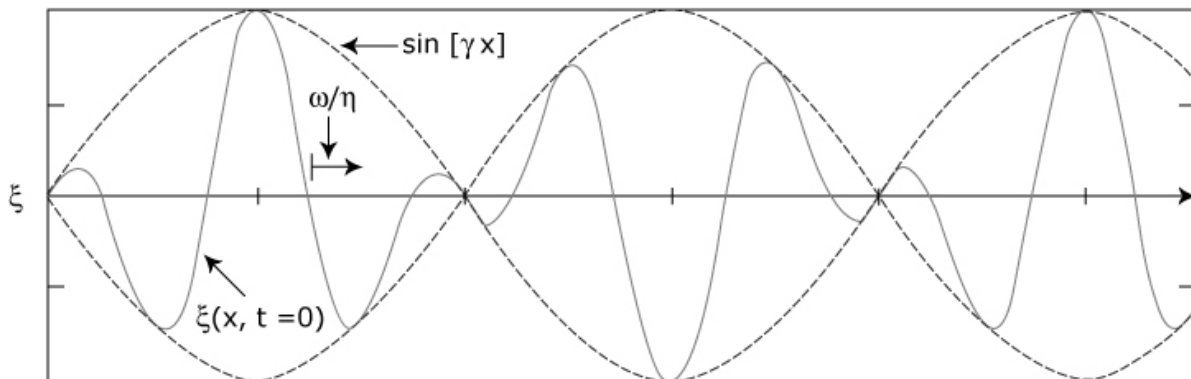
Dispersion relations for waves on a string moving with the equilibrium velocity  $U$  in the positive  $x$ -direction. Although the geometry for the subsonic and supersonic cases is the same as that for the characteristic lines in Figure on page ----, the axes here are  $\omega$ - $k$ , whereas in Figure on page ---- they are  $x$ - $t$ ; (a)  $U < v_s$ ; (b)  $U > v_s$ .

Supersonic Boundary Conditions ( $U > v_s$ ):  $\xi(0, t) = 0$

$$\frac{\partial \xi}{\partial x}(0, t) = \Delta_0 \cos \omega_d t$$

$$\xi(x, t) = \text{Re} \left[ A e^{j(\omega_d t - \eta x)} \sin \gamma x \right]$$

$$= \frac{\Delta_0}{\gamma} \cos(\omega_d t - \eta x) \sin \gamma x$$



Fast and slow waves interfere to form beats in space when  $U > v_s$ . The deflection is as the string appears if illuminated once each period of excitation  $2\pi/\omega_d$ .

### III. Evanescence and Oscillation with Convection

$$(\omega - kU)^2 = k^2 v_s^2 + \omega_c^2$$

$$\omega^2 - 2kU\omega + k^2 U^2 = k^2 v_s^2 + \omega_c^2$$

$$k^2 (U^2 - v_s^2) - 2kU\omega + \omega^2 - \omega_c^2 = 0$$

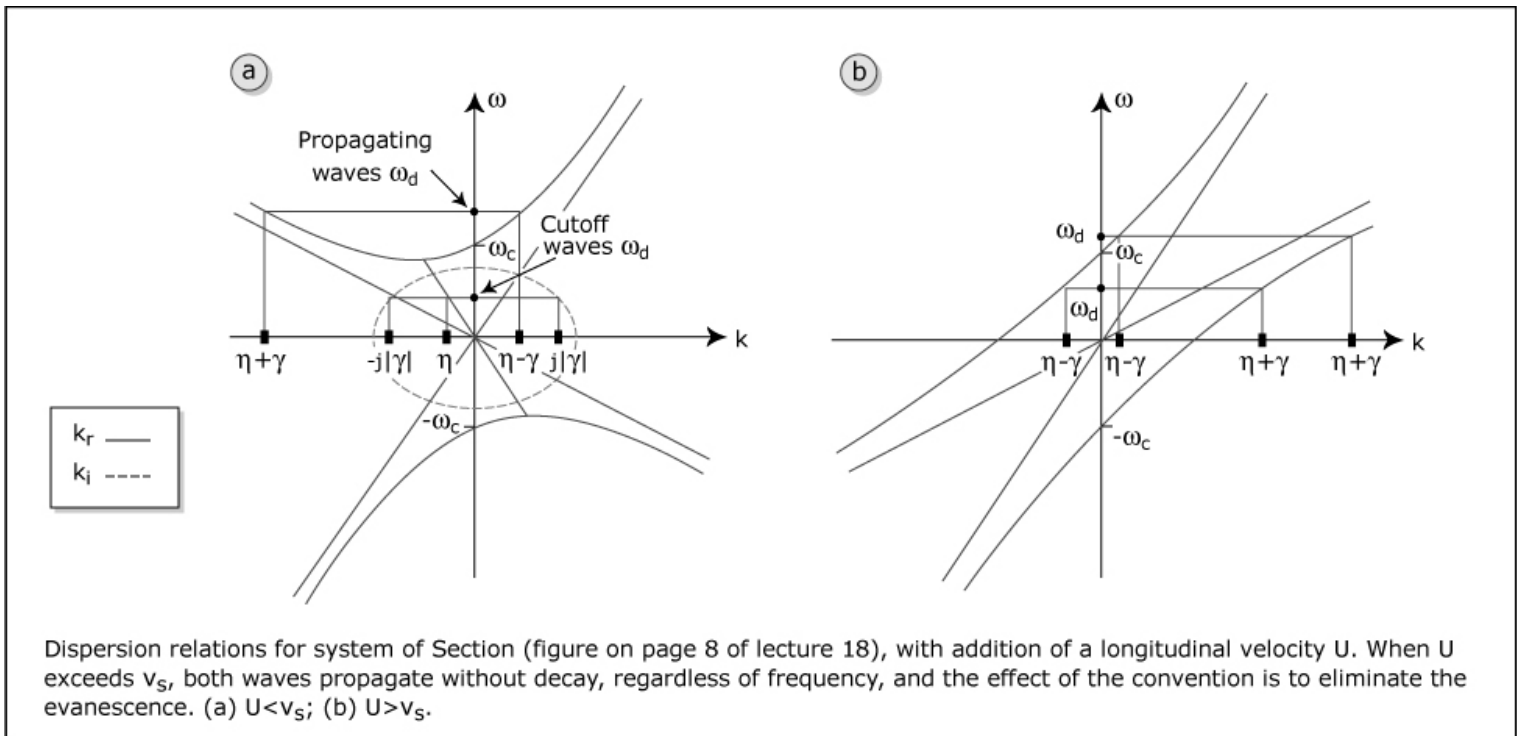
$$k = \frac{2\omega U \pm \sqrt{4\omega^2 U^2 - 4(U^2 - v_s^2)(\omega^2 - \omega_c^2)}}{2(U^2 - v_s^2)}$$

$$= \eta \pm \gamma$$

$$\eta = \frac{\omega U}{U^2 - v_s^2}, \quad \gamma = \frac{\sqrt{\omega^2 v_s^2 + (U^2 - v_s^2)\omega_c^2}}{(U^2 - v_s^2)}$$

If  $U > v_s$ ,  $\gamma$  real (Propagation without decay)

$U < v_s$ ,  $\gamma$  can be imaginary



#### IV. Convective Instability or Wave Amplification

$$(\omega - kU)^2 = k^2 v_s^2 - \omega_c^2$$

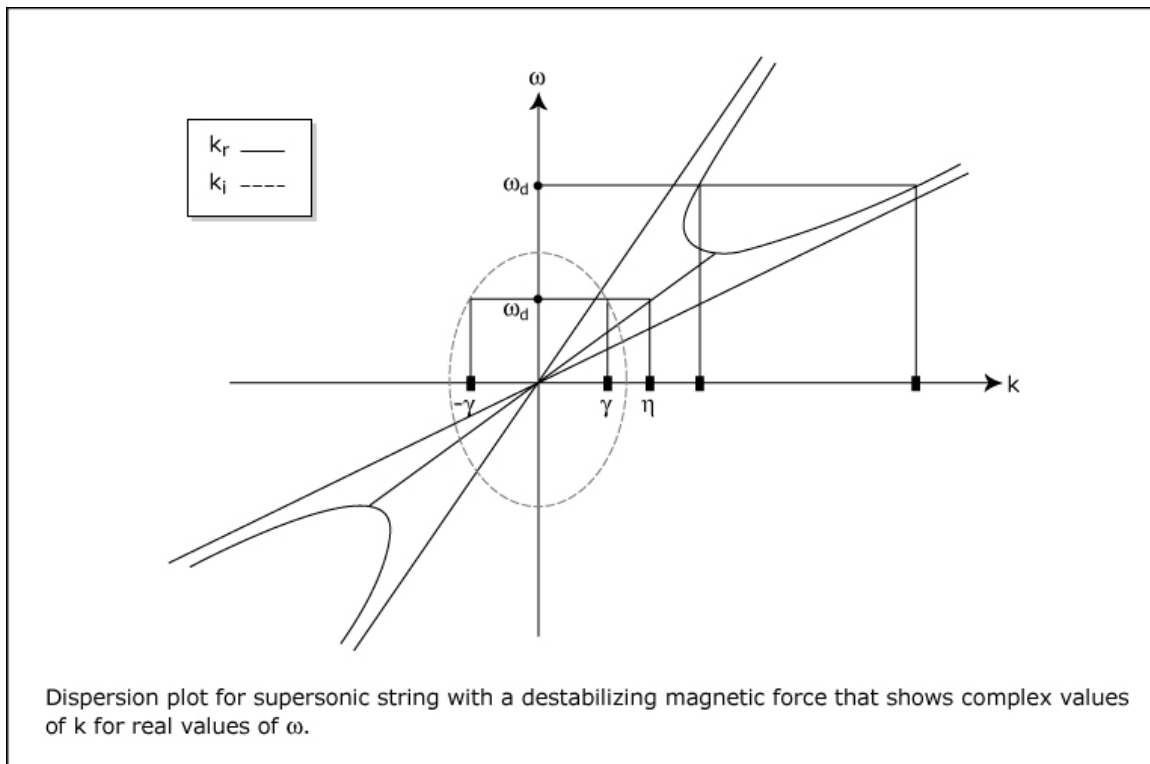
$$\omega = kU \pm \sqrt{k^2 v_s^2 - \omega_c^2}$$

$$k = \frac{\omega U \pm \sqrt{\omega^2 U^2 - (U^2 - v_s^2)(\omega^2 + \omega_c^2)}}{U^2 - v_s^2}$$

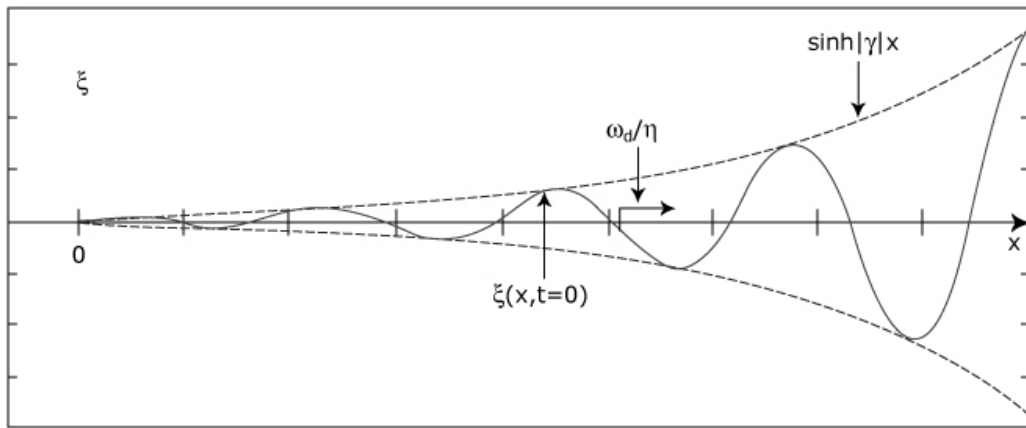
$$k = \eta \pm j\gamma$$

$$\eta = \frac{\omega U}{U^2 - v_s^2}, \quad \gamma = \frac{\sqrt{(U^2 - v_s^2)\omega_c^2 - \omega^2 v_s^2}}{U^2 - v_s^2}$$

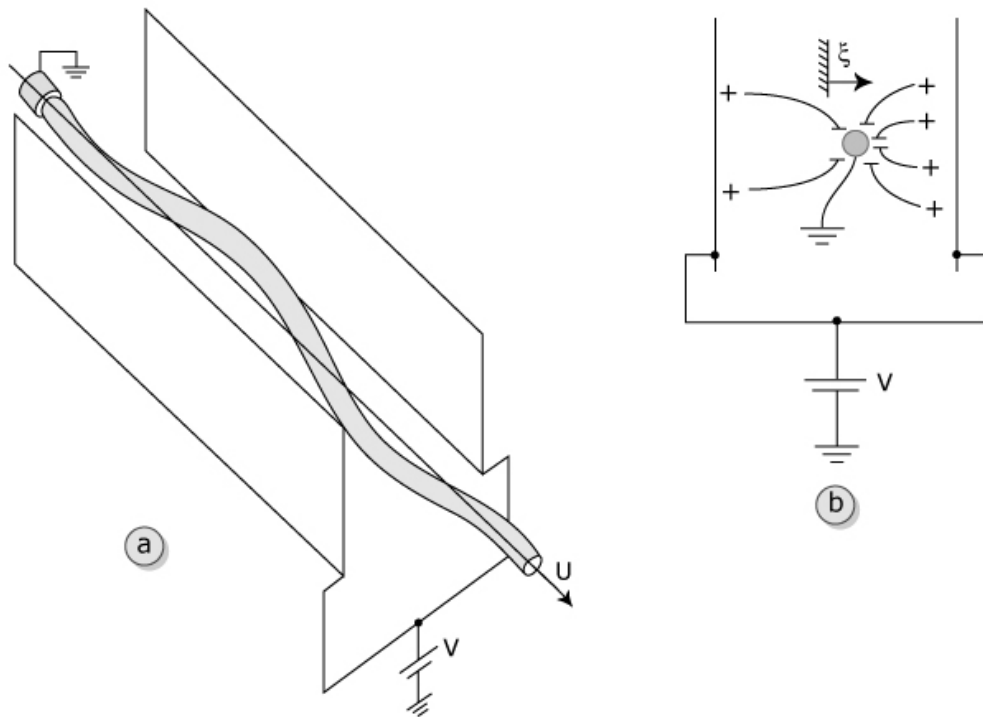
$$k \text{ complex } (\gamma \text{ real}) \text{ if } \omega^2 < \frac{(U^2 - v_s^2)}{v_s^2} \omega_c^2$$



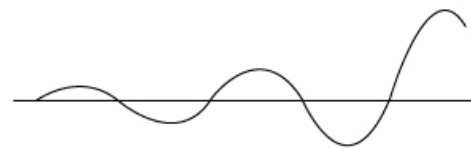
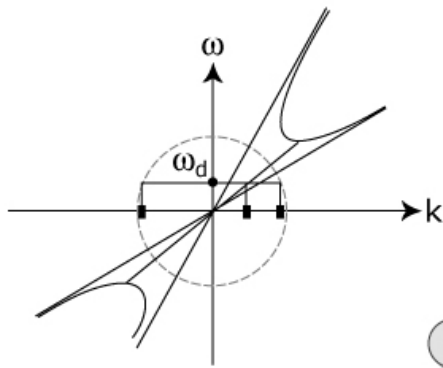
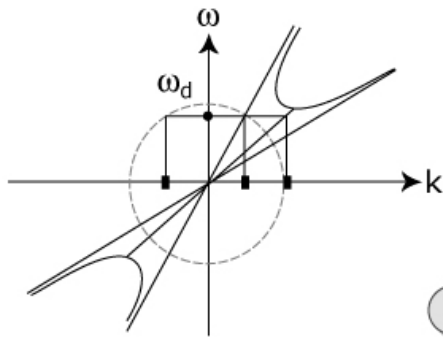
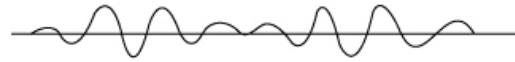
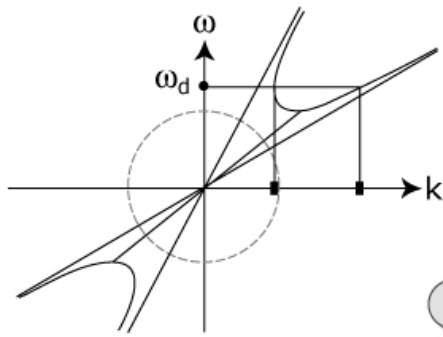
### Dynamics of Electromechanical Continua



Instantaneous view of string deflections when they are convectively unstable.



(a) A jet of water has a streaming velocity  $U$  midway between plane-parallel electrodes. A potential difference  $V$  is applied between the plates and the jet. (b) As the jet is deflected towards one of the plates, an unbalance in the electric force of attraction tends to carry it even farther in the direction of deflection. The instability resulting from this force is washed downstream to form the convective instability sketched in figure on page 7.



Sinusoidal excitation of jet shown in figure on page 6. Driving frequency decreases from (a) to (c). This experiment together with animated  $\omega$ - $k$  plots can be seen in the film "Complex Waves II," produced for the National Committee on Electrical Engineering Films by the Education Development Center, Newton, Mass.