# Summary of Signal Types

Typical Sets of Units

## Pulses:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ | $\leftrightarrow$ | $V(f)$ | [ V ] | $\leftrightarrow$ | $[VHz^{-1}]$ | [ V ] | $\leftrightarrow$ | $[VHz^{-1}]$ |
| $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ |
| $R(\tau)$ | $\leftrightarrow$ | $S(f) = \lvert \underline{V}(f) \rvert^2$ | $[V^2 s]$ | $\leftrightarrow$ | $[VHz^{-1}]^2$ | [J] | $\leftrightarrow$ | $[JHz^{-1}]$ |

1-ohm load

## Periodic:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $v(t)$ | $\leftrightarrow$ | $\underline{V}_m[v]$ | [ V ] | $\leftrightarrow$ | [V] | [ V ] | $\leftrightarrow$ | [ V ] |
| $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ |
| $R(\tau)$ | $\leftrightarrow$ | $\Phi_m = \lvert \underline{V}_m \rvert^2$ | $[V^2]$ | $\leftrightarrow$ | $[V^2]$ | [ W ] | $\leftrightarrow$ | [ W ] |

1-ohm load

## Random:

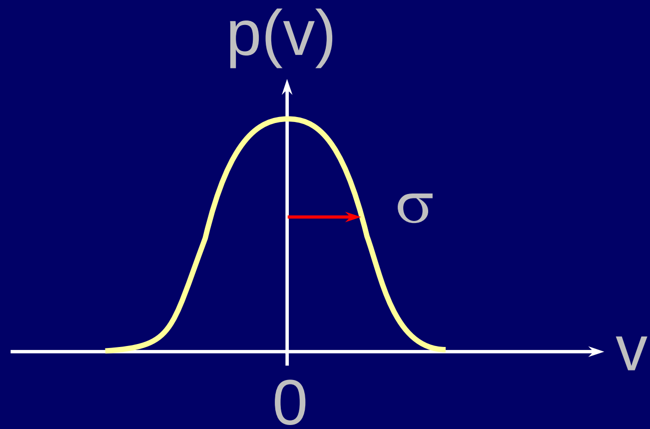
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $v(t)$ | $\leftrightarrow$ | [?] | [ V ] | $\leftrightarrow$ | [?] | [ V ] | $\leftrightarrow$ | [ ? ] |
| $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | | $\downarrow$ |
| $\Phi(\tau)$ | $\leftrightarrow$ | $\Phi(f)$ | $[V^2]$ | $\leftrightarrow$ | $[V^2Hz^{-1}]$ | [ W ] | $\leftrightarrow$ | [ W/Hz ] |

# Probability Distributions

Gaussian Noise:

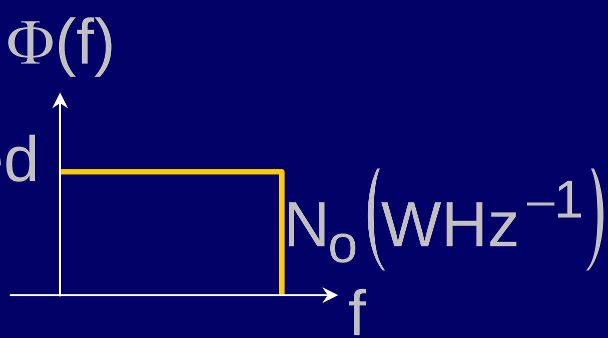
$$P\{v\} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(v/\sigma)^2/2}$$

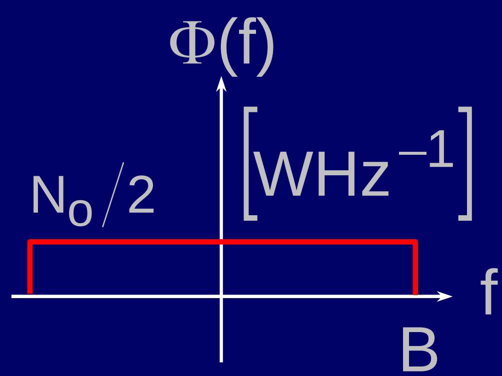
$$E\left[v^2\right] = \int_{-\infty}^{\infty} p(v)\, v^2 dv = \sigma^2$$




Band-limited Gaussian white noise, e.g. $N_o/2$

Single-sided equivalent




Power spectral density

$$E\left[v^2\right] = \sigma^2 = N_o B$$

$$\therefore \sigma = \sqrt{N_o B}$$

# Probability Distributions

Binomial Distribution:

Assume we have n bits, 0 or 1, where $p\{1\} \equiv p, p\{0\} \equiv 1-p$

$$p\{k\ 1's\} = \binom{n}{k} p^k (1-p)^{n-k} \text{ where } \binom{n}{k} \triangleq \frac{n!}{(n-k)!\,k!}$$

Note: There are n positions possible for the first "1," n – 1 for the second "1," and a total of n(n – 1)…(n – k + 1)/k! ways to arrange those k "1's" among the n available positions.

$$E[k] = np = \sum_{k=0}^{n} k\, p(k)$$

# Probability Distributions

$$E[k] = np = \sum_{k=0}^{n} k\, p(k)$$

Poisson Distribution:

Assume we have n bits, 0 or 1, where $p\{1\} \equiv p,\ p\{0\} \equiv 1 - p$

If $n >> 1,\ p << 1$: $np \overset{\Delta}{=} \lambda \cong 1$; variance $= np\underbrace{(1\text{-}p)}_{\sim 0} \cong \lambda$

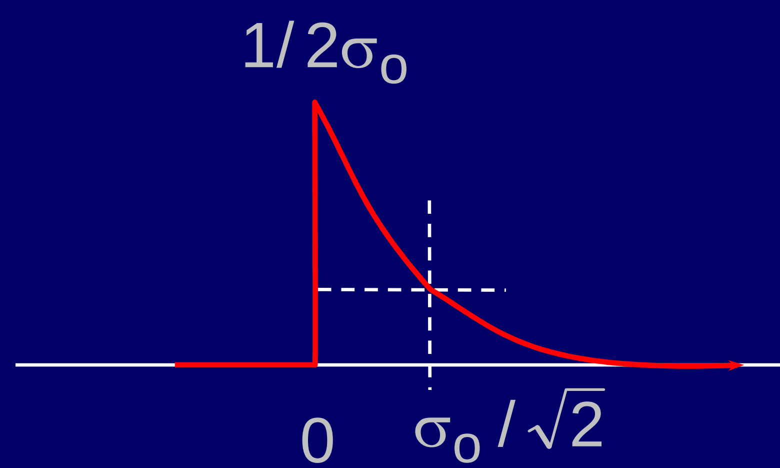
then $p\{k\} \cong \dfrac{\lambda^k}{k!} e^{-\lambda}$

Mean of k = $\lambda$ = np

Variance of k = $\lambda$

Laplacian Distribution:

$$p\{r\} = \frac{1}{2\sigma_o} e^{-\sqrt{2}|r|/\sigma_o}$$




$\left(\begin{array}{l}\text{Arises, for example, if } r^2 = x^2 + y^2,\ \overline{x^2} = \overline{y^2};\ \overline{xy} = 0 \\ \text{where x, y are Gaussian, variance } \sigma_o^2\text{, zero mean}\end{array}\right)$

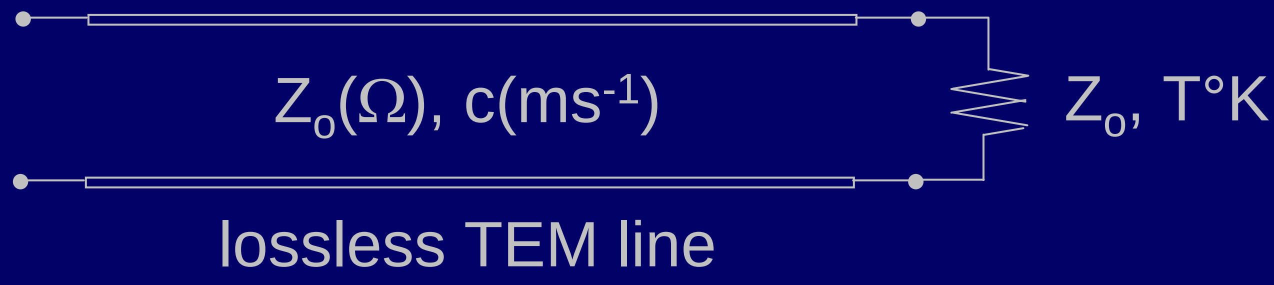
# Receiver-Noise Processes

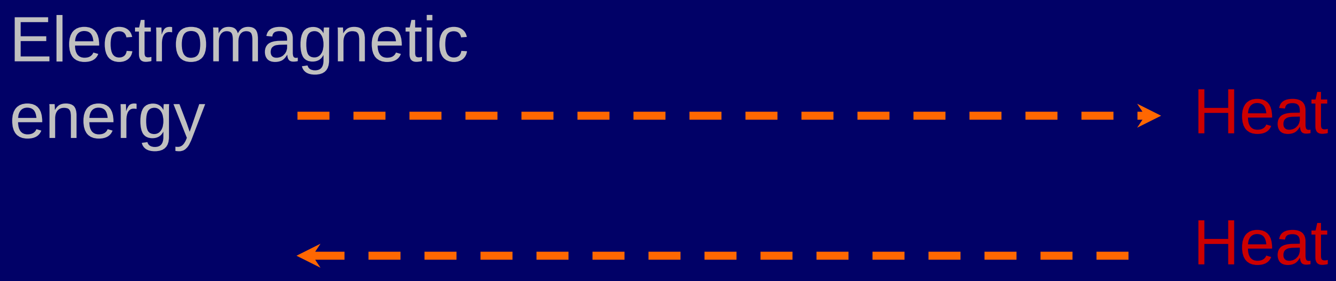
## Receivers are limited by noise, many types

## Thermal noise:

Cases: 1-D (TEM transmission line)
3-D (Multimode waveguide)
Equation of radiative transfer (1-D)
RF and optical limits; IR case

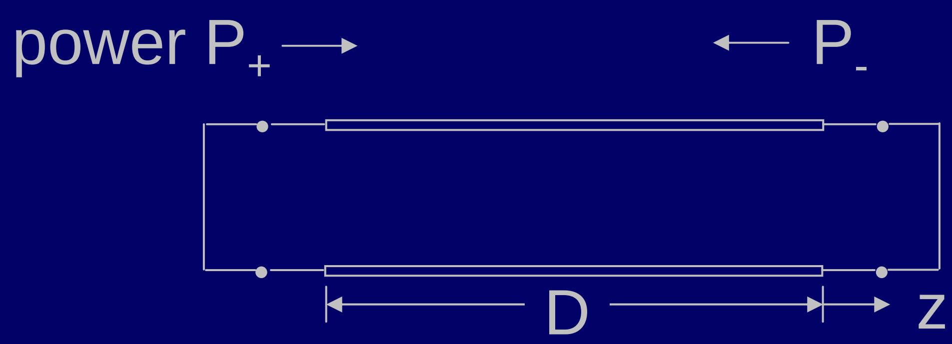
# Thermal noise, 1-D (TEM) case



# Approach:

closed container
very slightly lossy




1) Find average energy density W(f)[J/m Hz]
2) Find average power $P_+$[W/Hz] power flow

# Find average energy density $\overline{W}(f)[Jm^{-1}\ Hz^{-1}]$

$$\overline{W}(f) = \left(\frac{\text{modes}}{\text{Hz}}\right)\left(\frac{\text{photons}}{\text{mode}}\right)\underbrace{\left(\frac{\text{energy}}{\text{photon}}\right)}\bullet\frac{1}{D}$$

hf [Joules] (h ≡ Planck's constant)

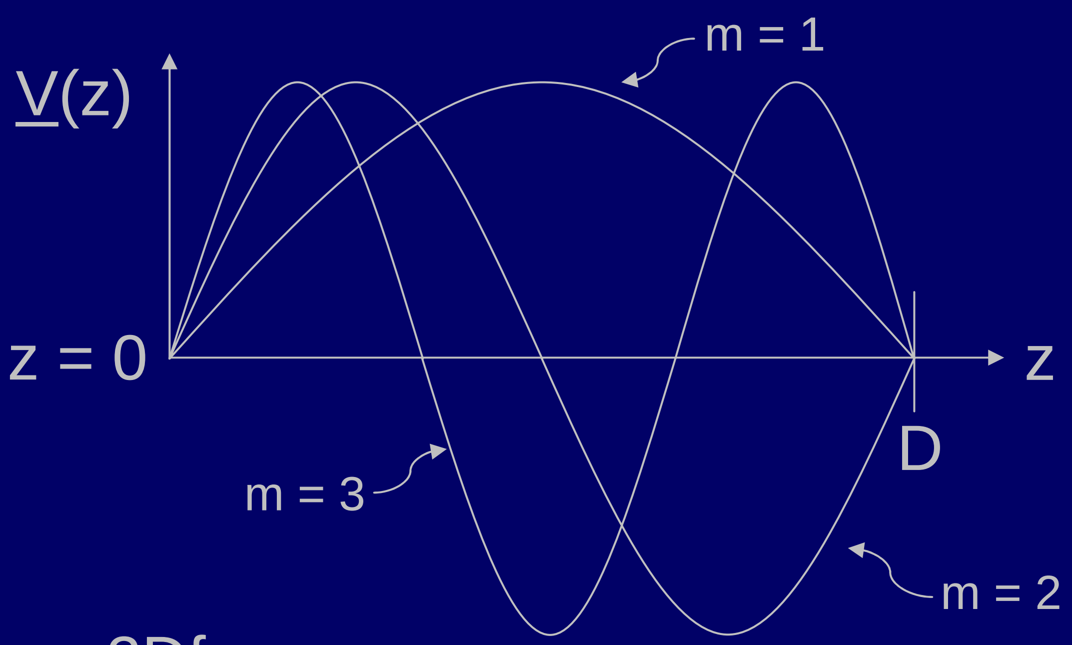
f is frequency (Hz)

$h = 6.6252 \times 10^{-34}$ (J s)

$$\overline{W}(f) = \left(\frac{\text{modes}}{\text{Hz}}\right)\left(\frac{\text{photons}}{\text{mode}}\right)\left(\frac{\text{energy}}{\text{photon}}\right) \bullet \frac{1}{D}$$

## Find modes/Hz:

Resonator modes




Therefore $m = \dfrac{2D}{\lambda_m} = \dfrac{2Df_m}{v_p}$ ($v_p$ = phase velocity)

$$\frac{dm}{df} = \frac{2D}{v_p} \quad \text{modes/Hz}$$

# Find photons/mode $\triangleq \bar{n}_j$; ($j^{th}$ mode)

Photons obey Bose-Einstein statistics; therefore any number can occupy each mode.

Total energy fixed; combinations favor more likely distributions

$$\bar{n}_j = \sum_{n=0}^{\infty} n\, p_j(n) \qquad p_j(n) \triangleq \left[ p\{n \text{ photons in state } j\} \right]$$

$$p_j(n) = Q\, e^{-nW_j/kT} \quad \text{, "Boltzmann distribution"}$$

$$\text{where } \sum_{n=0}^{\infty} p_j(n) \equiv 1,\ W_j \triangleq hf_j,\ Q = \text{constant}$$

$$p_j(n) = Q\, e^{-nW_j/kT}$$ , "Boltzmann distribution"

where $\sum_{n=0}^{\infty} p_j(n) \equiv 1$, $W_j \triangleq hf_j$, $Q$ = constant

$$\sum_{n=0}^{\infty} p_j(n) = Q \bullet \sum_{n=0}^{\infty} \left(e^{-W_j/kT}\right)^n = \frac{Q}{1 - e^{-W_j/kT}}$$

$$\left[\text{Recall } \sum_{n=0}^{\infty} x^n = 1/(1 - x) \text{ if } x < 1\right]$$

Therefore $Q = 1 - e^{-W_j/kT}$

## $p_j(n) = Q\, e^{-nW_j/kT}$ , "Boltzmann distribution"

Where $Q = 1 - e^{-W_j/kT}$

Therefore

$$p_j(n) = \left(1 - e^{-W_j/kT}\right) e^{-nW_j/kT}$$

$$\bar{n}_j = \sum_{n=0}^{\infty} n\, p_j(n) = \left(1 - e^{-W_j/kT}\right) \sum_{n=0}^{\infty} n \left(e^{-W_j/kT}\right)^n$$

$$\bar{n}_j = \sum_{n=0}^{\infty} n\, p_j(n) = \left(1 - e^{-W_j/kT}\right) \sum_{n=0}^{\infty} n \left(e^{-W_j/kT}\right)^n$$

Recall $\sum_{n=0}^{\infty} n\, x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} (1 - x)^{-1} = \frac{x}{(1 - x)^2}$

So $\bar{n}_j = \left(1 - e^{-W_j/kT}\right) \left[ e^{-W_j/kT} \Big/ \left(1 - e^{-W_j/kT}\right)^2 \right]$

$$\bar{n}_j = 1 \Big/ \left(e^{W_j/kT} - 1\right) \text{ photons/mode } \left[W_j = hf_j\right]$$

# Solution - Average Energy Density $\left(Jm^{-1}Hz^{-1}\right)$

$$\overline{W}(f) = \left(\frac{\text{modes}}{\text{Hz}}\right)\left(\frac{\text{photons}}{\text{mode}}\right)\left(\frac{\text{energy}}{\text{photon}}\right) \bullet \frac{1}{D}$$

$$W(f) = \left(\frac{2D}{v_p}\right)\left(\frac{1}{e^{W_j/kT} - 1}\right)(hf) \bullet \frac{1}{D} = \frac{2hf}{v_p\left(e^{hf/kT} - 1\right)} \quad \left[Jm^{-1}Hz^{-1}\right]$$

$$W(f) = W_+ + W_- = 2W_+$$

(powers and energies superimpose if waves are "orthogonal")

$W_+$ = forward-moving energy density

# Solution - Thermal power in TEM line:

$$W(f) = \left(\frac{2D}{v_p}\right)\left(\frac{1}{e^{W_j/kT} - 1}\right)(hf) \bullet \frac{1}{D} = \frac{2hf}{v_p\left(e^{hf/kT} - 1\right)} \left[Jm^{-1}Hz^{-1}\right]$$

$$W(f) = W_+ + W_- = 2W_+$$

$$P_+\left[WH_z^{-1}\right] = v_g W_+ = v_g W/2$$

If the TEM line is non-dispersive, then $v_p = v_g$ and

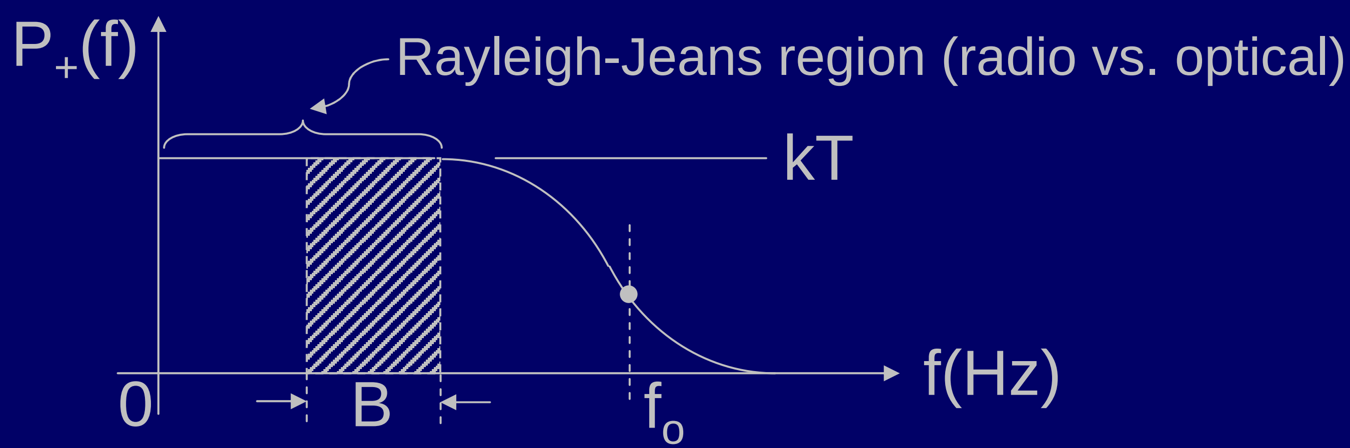
$$P_+(f)\left[WH_z^{-1}\right] = \frac{hf}{e^{hf/kT} - 1}$$

Recall $e^x = 1 + x + x^2/2! + \ldots \approx 1 + x$ for $x << 1$

$$P_+(f)\left[WHz^{-1}\right] = \frac{hf}{e^{hf/kT} - 1} \cong kT \text{ for } hf << kT$$

"Rayleigh-Jeans limit"

$P \cong kTB$ watts in uniform bandwidth B(Hz)

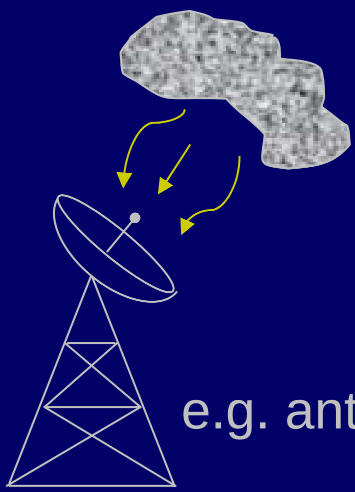



$hf_o \cong kT$, so $f_o = kT/h \cong 20 \bullet T(°K)$ GHz

Planck's constant: $h \cong 6.6 \times 10^{-34}$ [J sec]
Boltzmann's constant: $k = 1.38 \times 10^{-23}$ [J/°K]

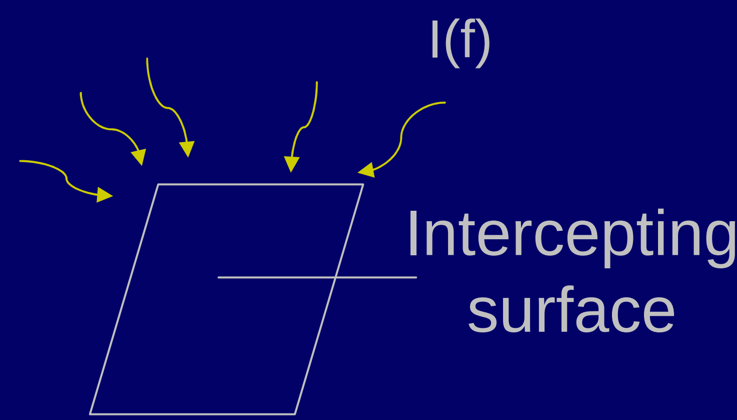
# Problem: Find thermal radiation intensity I(watts/Hz • m$^2$ • ster)

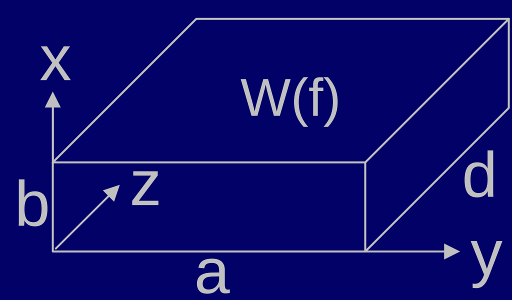
Approach: Assume closed container, very slightly lossy, filled with photons



e.g. antenna

1) Find energy density spectrum W(f)[J/m$^3$ Hz]
2) Relate W(f) to I(f)



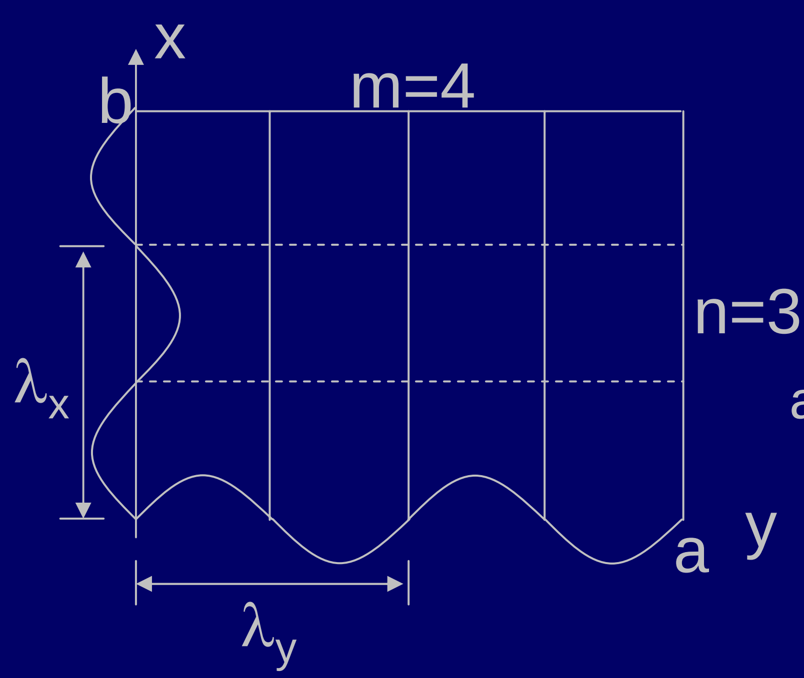
# First:

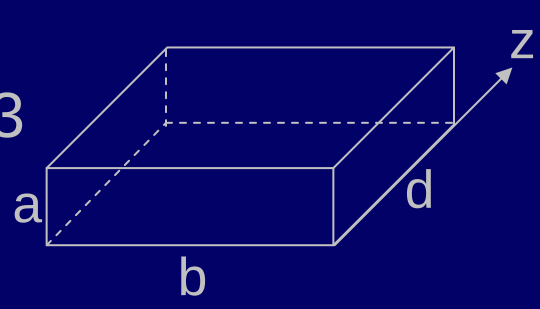
# Find energy density spectrum W(f)[Jm$^{-3}$Hz$^{-1}$]

$$W(f) = \left(\frac{\text{modes}}{\text{Hz}}\right) \bullet \left(\frac{\text{photons}}{\text{mode}}\right) \bullet \left(\frac{\text{energy}}{\text{photon}}\right) \bullet \frac{1}{\text{vol.}}$$

| modes/Hz | photons/mode | energy/photon |
|---|---|---|
| waveguide modes: $\underbrace{TE_{m,n}}_{E_z \equiv 0}$, $\underbrace{TM_{m,n}}_{H_z \equiv 0}$ | $1 / \left(e^{hf/kT} - 1\right)$ | $hf$ |

$$\frac{m\lambda_y}{2} = a, \quad \frac{n\lambda_x}{2} = b$$



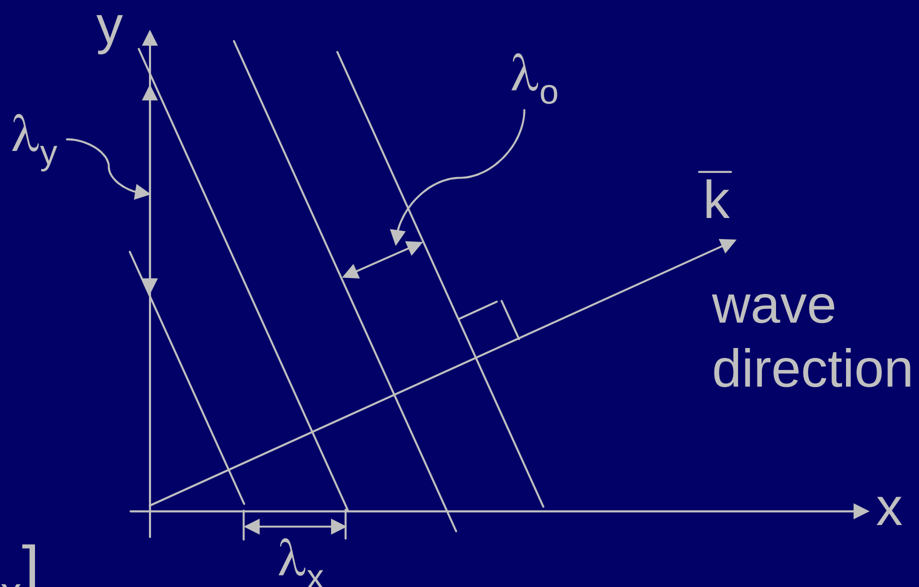



# Begin finding modes/Hz

Claim: $$f_{m,n,p} = \sqrt{\left(\frac{cm}{2a}\right)^2 + \left(\frac{cn}{2b}\right)^2 + \left(\frac{cp}{2d}\right)^2}$$

Recall wave eqn: $\left[\nabla^2 + \omega^2\mu\varepsilon\right]\overline{E} = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$



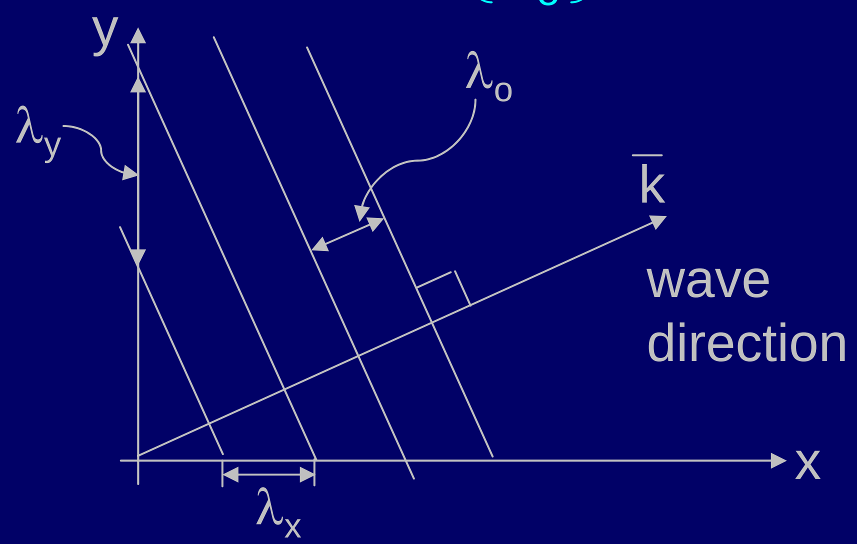

$\overline{E} = \overline{E}_o e^{-jk_x x - jk_y y - jk_z z}$ $\quad [k_x = 2\pi/\lambda_x]$

$$\Rightarrow k_x^2 + k_y^2 + k_z^2 = k_o^2 = \omega^2\mu_o\varepsilon_o = \left(\frac{2\pi}{\lambda_o}\right)^2$$

# Uniform plane wave

$\underline{\overline{E}} = \underline{\overline{E}}_o e^{-jk_x x - jk_y y - jk_z z}$ $\quad [k_x = 2\pi/\lambda_x]$ $\quad \left(\nabla^2 + \omega^2 \mu\varepsilon\right)\underline{\overline{E}} = 0$

$$k_x^2 + k_y^2 + k_z^2 = k_o^2 = \omega^2 \mu_o \varepsilon_o = \left(\frac{2\pi}{\lambda_o}\right)^2$$



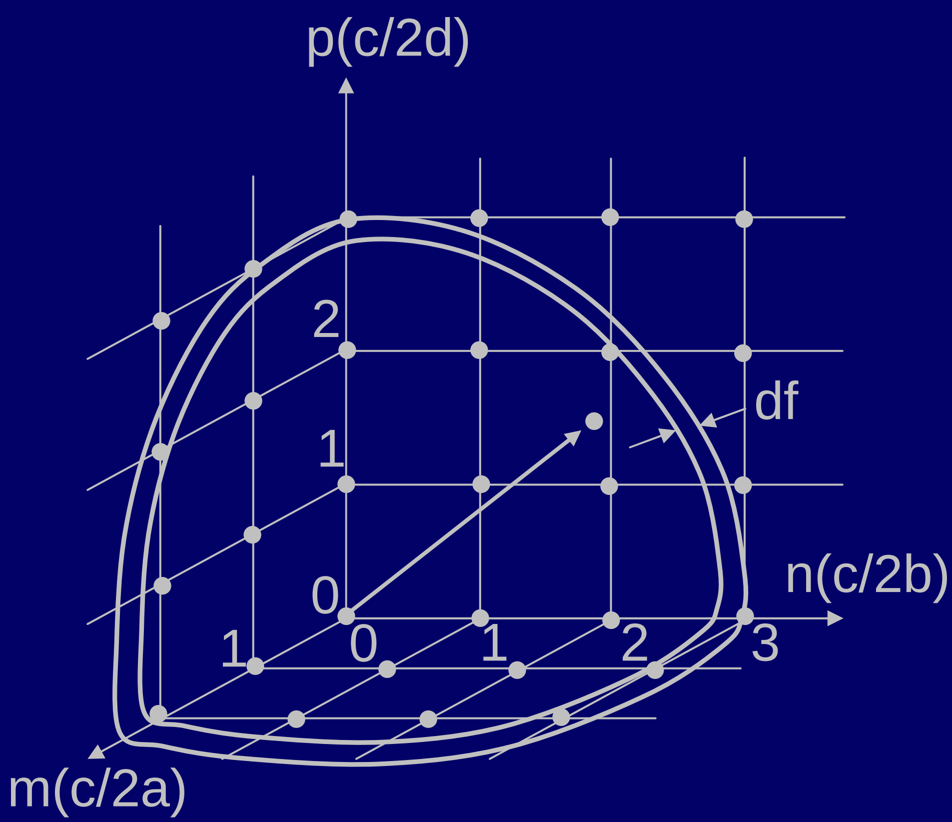

Therefore $1/\lambda_x^2 + 1/\lambda_y^2 + 1/\lambda_z^2 = 1/\lambda_o^2 = (f/c)^2$

Note: $\quad m\frac{\lambda_y}{2} = a \Rightarrow (m/2a)^2 + (n/2b)^2 + (p/2d)^2 = (f/c)^2 = \text{QED}$

Next, use this relation to find modes/Hz

# Find modes/Hz:

$$f_{m,n,p} = \sqrt{\left(\frac{cm}{2a}\right)^2 + \left(\frac{cn}{2b}\right)^2 + \left(\frac{cp}{2d}\right)^2}$$




\# modes in df shell =

= vol. of shell × 2/vol. cell

↓

TE + TM

$$= \frac{4\pi f^2 df \cdot 2}{8} \bigg/ \left(\frac{c}{2a} \cdot \frac{c}{2b} \cdot \frac{c}{2c}\right)$$

$$= \frac{8\pi f^2}{c^3} V_{ol}\, df \Rightarrow \left[\frac{\text{modes}}{\text{Hz}}\right] df$$

$V_{ol}$ ↓ abd

# Find energy density spectrum W(f):

$$W(f) = \left(\frac{\text{modes}}{\text{Hz}}\right) \bullet \left(\frac{\text{photons}}{\text{mode}}\right) \bullet \left(\frac{\text{energy}}{\text{photon}}\right) \bullet \frac{1}{\text{vol.}}$$

$$W(f) = \left(\frac{8\pi f^2}{c^3} V\right)\left(\frac{1}{e^{hf/kT} - 1}\right) \bullet hf \bullet 1/V = \frac{8\pi}{c^3} \frac{hf^3}{e^{hf/kT} - 1} \; [\text{Jm}^{-3}\text{Hz}^{-1}]$$