Prof. P.A. Parrilo Spring 2006

MIT 6.972
Algebraic methods and semidefinite programming
Homework assigment # 3

Date Given: May 12th, 2006
Date Due: May 19th, 12PM

P1. [***30 pts***] Recall the relaxations for linearly and quadratically constrained quadratic programming we have seen earlier (concretely, equation (9) in Lecture 3). Explain how these can be interpreted as a special case of Positivstellensatz-based relaxations (and more specifically, a Schmüdgen-type certificate).

P2. [***35 pts***] In this problem, we analyze symmetry reduction in the case of sum of squares decompositions of univariate even polynomials. Let $p(x)$ be a univariate polynomial that satisfies $p(x) = p(-x)$ (i.e., it is even).

- **(a)** Write down the "standard" SDP formulation for checking whether $p(x)$ is SOS.
- **(b)** Is this SDP invariant under the action of a group?
- **(c)** Restrict the feasible set to the fixed-point subspace. How does the problem simplify? (Hint: you may want to group the monomials depending on whether the exponents are even or odd).
- **(d)** Explain why the new formulation is computationally better.
- **(e)** Compare the results with making the substitution $y = x^2$ in the original polynomial, and imposing the constraint $y \geq 0$. How do they differ (if they do)?

P3. [***35 pts***] Let $M \in \mathcal{S}^n$, and let $z = [x_1^2, \ldots, x_n^2]^T$. As we have seen, M is copositive if and only if the homogeneous quartic polynomial $p(x) = z^T M z$ is nonnegative.

- **(a)** Plot the region of $(a, b) \in \mathbb{R}^2$ for which the matrix

$$\begin{bmatrix} a & b \\ b & 1 \end{bmatrix}$$

is copositive.
- **(b)** Prove that $p(x)$ is a sum of squares if and only if $M = P + N$, where P is positive semidefinite and N is componentwise nonnegative. (Hint: you may want to use the symmetry $p(x_1, \ldots, x_n) = p(\pm x_1, \ldots, \pm x_n)$).