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6.976
High Speed Communication Circuits and Systems
Lecture 2
Transmission Lines

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Maxwell's Equations

- General form:

$$\nabla \times E = -\mu \frac{dH}{dt} \qquad (1)$$

$$\nabla \times H = J + \epsilon \frac{dE}{dt} \qquad (2)$$

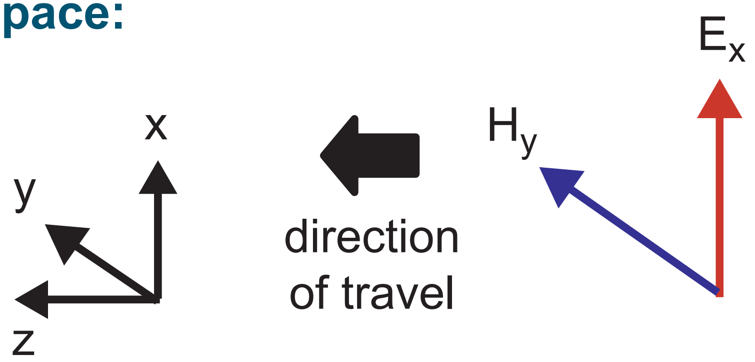
$$\nabla \cdot \epsilon E = \rho \qquad (3)$$

$$\nabla \cdot \mu H = 0 \qquad (4)$$

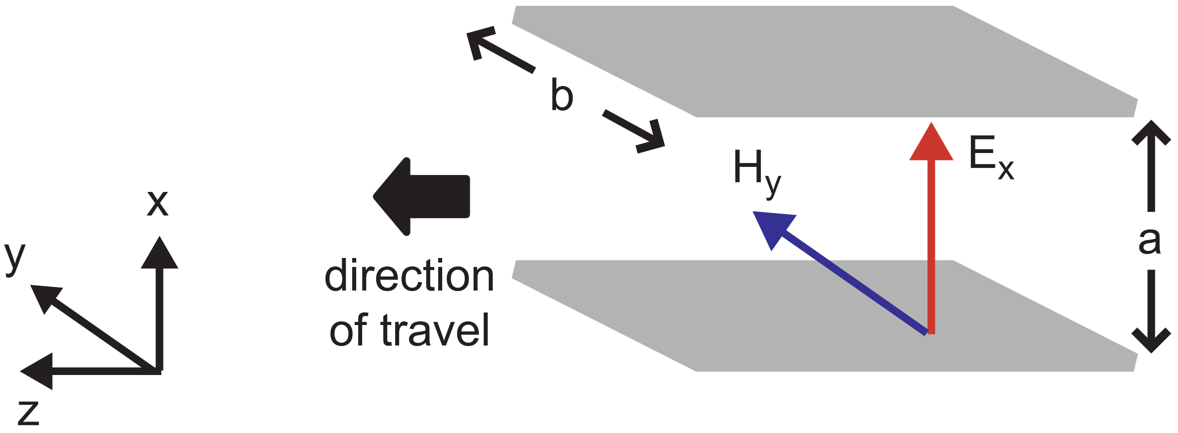
- Assumptions for free space and transmission line propagation
  - No charge buildup ⇒ $\rho = 0$
  - No free current ⇒ $J = 0$
- Note: we'll only need Equations 1 and 2

Assumptions

- Orientation and direction
  - E field is in x-direction and traveling in z-direction
  - H field is in y-direction and traveling in z-direction
  - In freespace:



- For transmission line (TEM mode)



Solution

- Fields change only in time and in z-direction
  - Assume complex exponential solution

$$E = \hat{x}E_x(z,t) = \hat{x}E_o e^{-jkz} e^{jwt}$$

$$H = \hat{y}H_y(z,t) = \hat{y}H_o e^{-jkz} e^{jwt}$$

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- Implications:

$$\frac{dE_x(z,t)}{dz} = -jkE_x(z,t), \quad \frac{dE_x(z,t)}{dt} = jwE_x(z,t)$$

$$\frac{dH_y(z,t)}{dz} = -jkH_y(z,t), \quad \frac{dH_y(z,t)}{dt} = jwH_y(z,t)$$

Solution

- **Fields change only in time and in z-direction**
  - **Assume complex exponential solution**

$$E = \hat{x}E_x(z,t) = \hat{x}E_o e^{-jkz}e^{jwt}$$

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- **Implications:**

$$\frac{dE_x(z,t)}{dz} = -jkE_x(z,t), \quad \frac{dE_x(z,t)}{dt} = jwE_x(z,t)$$

$$\frac{dH_y(z,t)}{dz} = -jkH_y(z,t), \quad \frac{dH_y(z,t)}{dt} = jwH_y(z,t)$$

But, what is the value of k ?

Evaluate Curl Operations in Maxwell's Formula

- **Definition**

$$\nabla \times E = \widehat{x}\left(\frac{dE_z}{dy} - \frac{dE_y}{dz}\right) + \widehat{y}\left(\frac{dE_x}{dz} - \frac{dE_z}{dx}\right) + \widehat{z}\left(\frac{dE_y}{dx} - \frac{dE_x}{dy}\right)$$

$$\nabla \times H = \widehat{x}\left(\frac{dH_z}{dy} - \frac{dH_y}{dz}\right) + \widehat{y}\left(\frac{dH_x}{dz} - \frac{dH_z}{dx}\right) + \widehat{z}\left(\frac{dH_y}{dx} - \frac{dH_x}{dy}\right)$$

Evaluate Curl Operations in Maxwell's Formula

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$$\nabla \times H = \hat{x}\left(\frac{dH_z}{dy} - \frac{dH_y}{dz}\right) + \hat{y}\left(\frac{dH_x}{dz} - \frac{dH_z}{dx}\right) + \hat{z}\left(\frac{dH_y}{dx} - \frac{dH_x}{dy}\right)$$

- **Given the previous assumptions**

$$\nabla \times E = \hat{y}\,\frac{dE_x(z,t)}{dz} = -\hat{y}\; jkE_x(z,t)$$

$$\nabla \times H = -\hat{x}\,\frac{dH_y(z,t)}{dz} = \hat{x}\; jkH_y(z,t)$$

Now Put All the Pieces Together

- **Solve Maxwell's Equation (1)**

$$\nabla \times E = -\mu \frac{dH}{dt} \Rightarrow -\widehat{y}\ jkE_x(z,t) = -\widehat{y}\ \mu j w H_y(z,t)$$

$$\Rightarrow \frac{E_x(z,t)}{H_y(z,t)} = \frac{\mu w}{k} \quad \text{(intrinsic impedance)}$$

Now Put All the Pieces Together

- **Solve Maxwell's Equations (1) and (2)**

$$\nabla \times E = -\mu \frac{dH}{dt} \Rightarrow -\hat{y}\; jkE_x(z,t) = -\hat{y}\; \mu j w H_y(z,t)$$

$$\Rightarrow \frac{E_x(z,t)}{H_y(z,t)} = \frac{\mu w}{k} \quad \text{(intrinsic impedance)}$$

$$\nabla \times H = \epsilon \frac{dE}{dt} \Rightarrow \hat{x}\; jkH_y(z,t) = \hat{x}\; \epsilon j w E_x(z,t)$$

$$\Rightarrow H_y(z,t) = \frac{\epsilon w}{k} E_x(z,t) = \frac{\epsilon w}{k}\left(\frac{\mu w}{k}\right) H_y(z,t)$$

$$\Rightarrow \frac{\epsilon w}{k}\left(\frac{\mu w}{k}\right) = 1 \Rightarrow \boxed{k = w\sqrt{\mu\epsilon}}$$

Now Put All the Pieces Together

- **Solve Maxwell's Equations (1) and (2)**

$$\nabla \times E = -\mu \frac{dH}{dt} \;\Rightarrow\; -\hat{y}\; jkE_x(z,t) = -\hat{y}\; \mu jwH_y(z,t)$$

$$\Rightarrow\; \frac{E_x(z,t)}{H_y(z,t)} = \frac{\mu w}{k} \quad \text{(intrinsic impedance)}$$

$$\nabla \times H = \epsilon \frac{dE}{dt} \;\Rightarrow\; \hat{x}\; jkH_y(z,t) = \hat{x}\; \epsilon jwE_x(z,t)$$

$$\Rightarrow\; H_y(z,t) = \frac{\epsilon w}{k} E_x(z,t) = \frac{\epsilon w}{k}\left(\frac{\mu w}{k}\right) H_y(z,t)$$

$$\Rightarrow\; \frac{\epsilon w}{k}\left(\frac{\mu w}{k}\right) = 1 \;\Rightarrow\; k = w\sqrt{\mu\epsilon}$$

$$\boxed{\Rightarrow\; \text{intrinsic impedance} = \frac{\mu w}{k} = \frac{\mu w}{w\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}}$$

Connecting to the Real World

- **Current solution is complex**

$$E = \widehat{x}E_x(z,t) = \widehat{x}E_o e^{-jkz}e^{jwt} = \widehat{x}E_o e^{-j(wt-kz)}$$

- **But the following complex solution is also valid**

$$E = \widehat{x}E_x(z,t) = \widehat{x}E_o e^{j(wt-kz)}$$

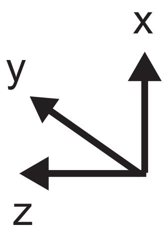
- **And adding them together is also a valid solution that is now real-valued**

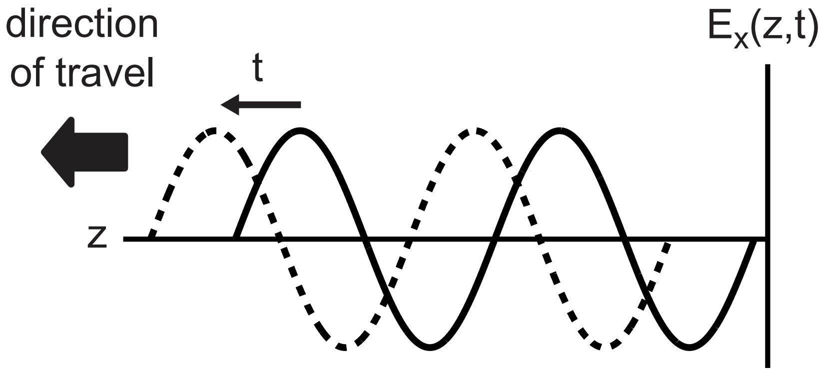
$$E = \widehat{x}\ E_o(e^{j(wt-kz)} + e^{-j(wt-kz)})$$

$$= \widehat{x}\ 2E_o \cos(wt - kz)$$

Calculating Propagation Speed

- **The resulting cosine wave is a function of time AND position**





$$E_x(z,t) = \hat{x}\; 2E_o \cos(wt - kz)$$

- **Consider "riding" one part of the wave**

$$-kz + wt = \text{constant (choose 0)} \;\Rightarrow\; z = \frac{wt}{k}$$

- **Velocity calculation**

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{wt}{k}\right) = \frac{w}{k} = \frac{w}{w\sqrt{\mu\epsilon}} = \boxed{\frac{1}{\sqrt{\mu\epsilon}}}$$

Freespace Values

- **Constants**

$$\epsilon = \epsilon_o = \frac{1}{36\pi} \times 10^{-9} \text{ F/m}$$

$$\mu = \mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

- **Impedance**

$$\sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{\epsilon_o}} = \text{ 377 Ohms}$$

- **Propagation speed**

$$\frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_o\epsilon_o}} = 30 \times 10^9 \text{ cm/s}$$

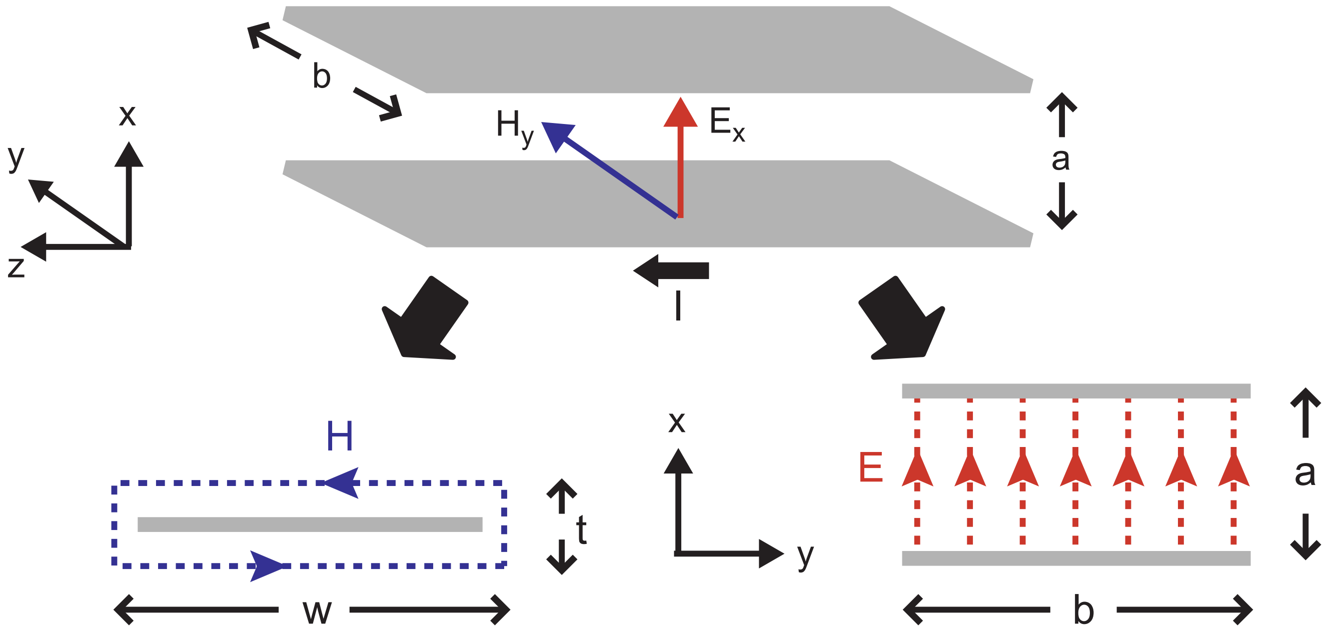
- **Wavelength of 30 GHz signal**

$$\lambda = \frac{T}{\sqrt{\mu\epsilon}} = \frac{1}{f\sqrt{\mu_o\epsilon_o}} = \text{ 1 cm}$$

Voltage and Current

- **Definitions:** $V = \int_{C_t} E \cdot dl \quad \text{(path integral)}$

$$I = \oint_{C_o} H \cdot dl \quad \text{(contour integral)}$$

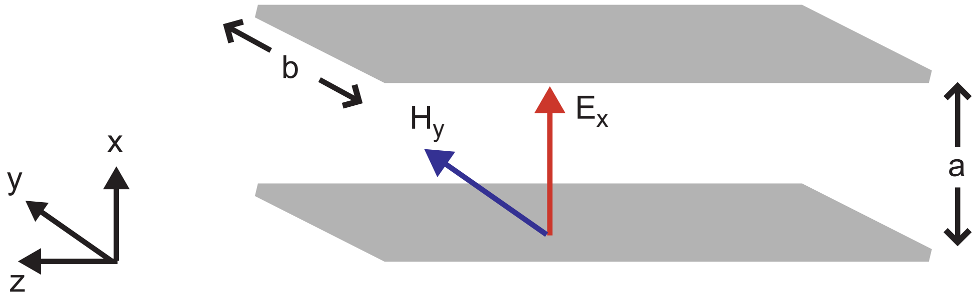


$$I = (2\text{w} + 2\text{t})H$$

$$V = aE$$

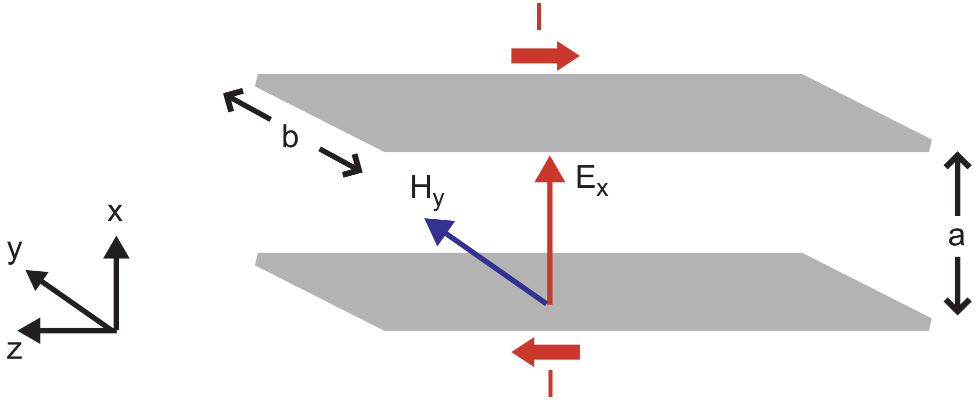
Parallel Plate Waveguide

- **E-field and H-field are influenced by plates**



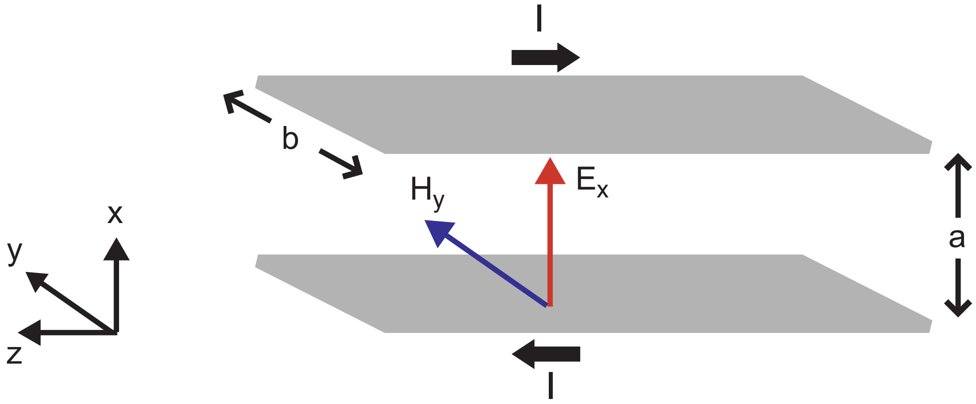
Current and H-Field

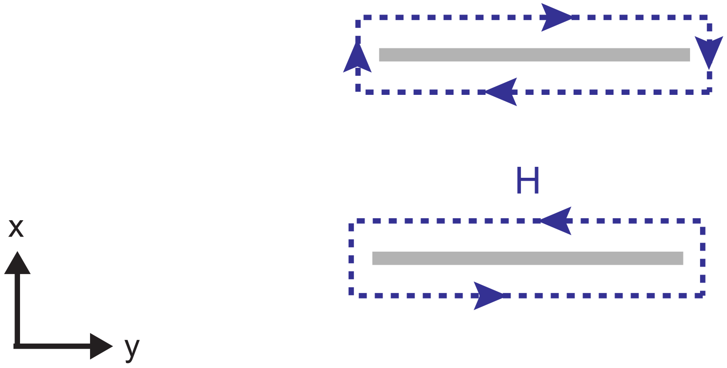
- **Assume that (AC) current is flowing**



Current and H-Field

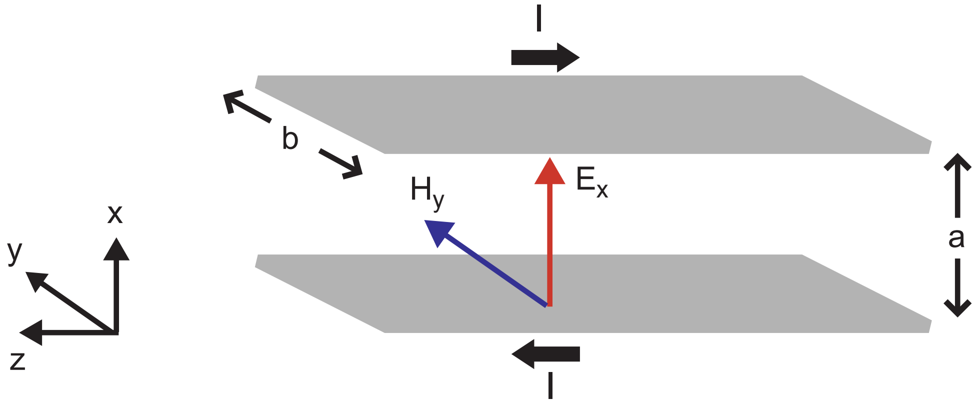
- **Current flowing down waveguide influences H-field**

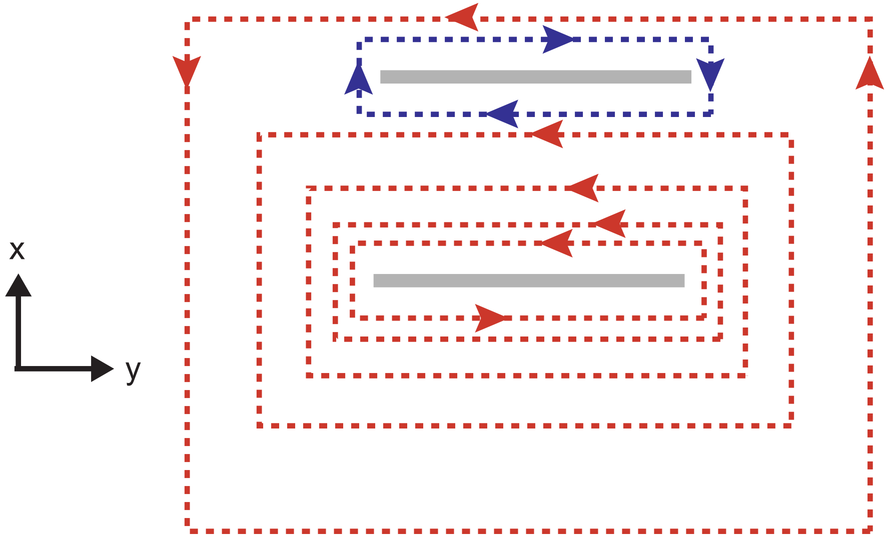




Current and H-Field

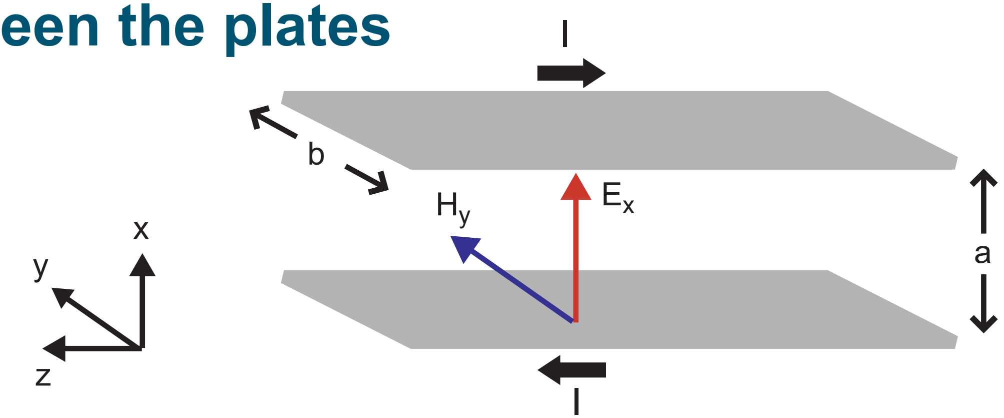
- **Flux from one plate interacts with flux from the other plate**

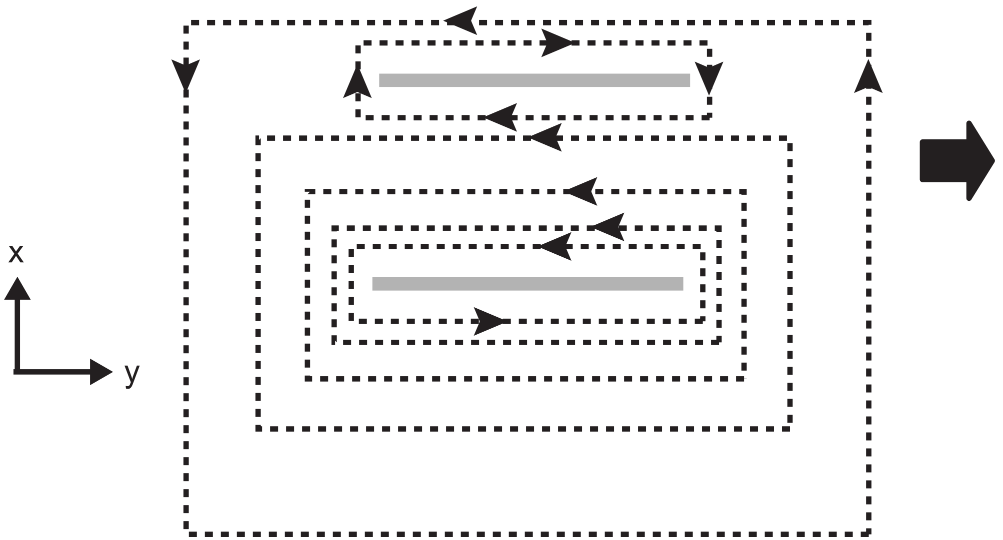


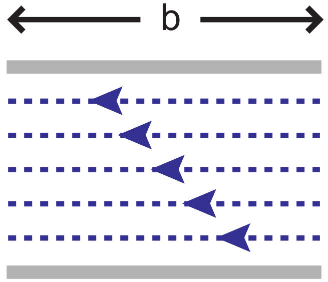


Current and H-Field

- **Approximate H-Field to be uniform and restricted to lie between the plates**



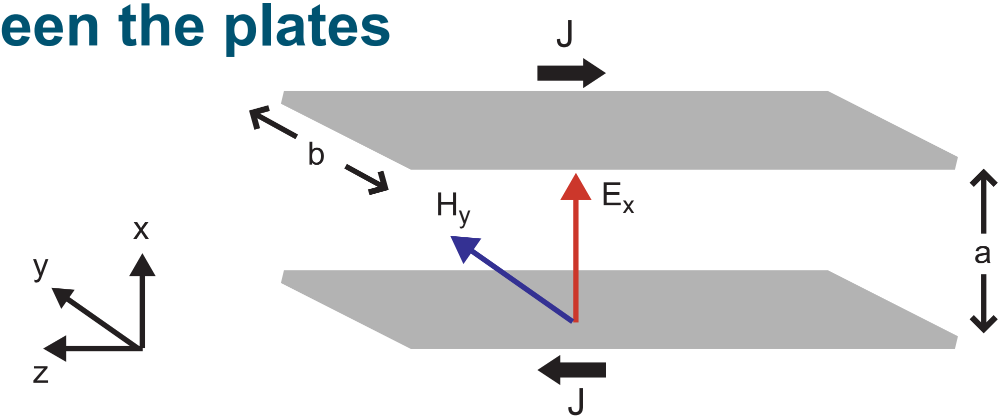


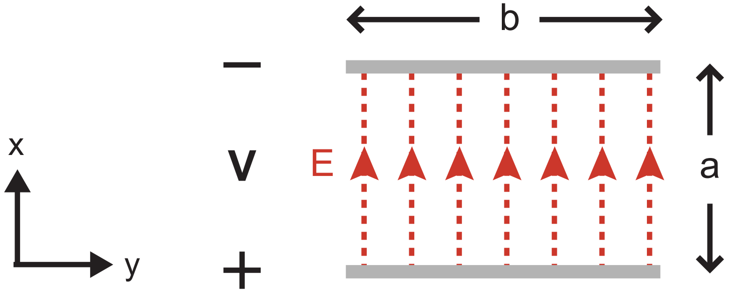


$$I = bH$$

Voltage and E-Field

- **Approximate E-field to be uniform and restricted to lie between the plates**





$$V = aE$$

Back to Maxwell's Equations

- **From previous analysis**

$$\nabla \times E = -\mu \frac{dH}{dt} \;\Rightarrow\; jkE_x(z,t) = jw\mu H_y(z,t)$$

$$\nabla \times H = \epsilon \frac{dE}{dt} \;\Rightarrow\; jkH_y(z,t) = jw\epsilon E_x(z,t)$$

- **These can be equivalently written as**

$$jk\left(aE_x(z,t)\right) = jw\mu \frac{a}{b}\left(bH_y(z,t)\right) \;\Rightarrow\; jkV(z,t) = jwLI(z,t)$$

$$jk(bH_y(z,t)) = jw\epsilon \frac{b}{a}(aE_x(z,t)) \;\Rightarrow\; jkI(z,t) = jwCV(z,t)$$

- **Where**

$$L = \mu \frac{a}{b}$$ (inductance per unit length - H/m)

$$C = \epsilon \frac{b}{a}$$ (capacitance per unit length - F/m)

Wave Equation for Transmission Line (TEM)

- **Key formulas**

$$jkV(z,t) = jwLI(z,t) \qquad (1)$$

$$jkI(z,t) = jwCV(z,t) \qquad (2)$$

- **Substitute (2) into (1)**

$$jkV(z,t) = jwL\left(\frac{w}{k}CV(z,t)\right) \Rightarrow (k^2 - w^2LC)V(z,t) = 0$$

$$\boxed{\Rightarrow \; k = w\sqrt{LC}}$$

- **Characteristic impedance (use Equation (1))**

$$\frac{V(z,t)}{I(z,t)} = \frac{wL}{k} = \frac{wL}{w\sqrt{LC}} = \boxed{\sqrt{\frac{L}{C}}}$$

Connecting to the Real World

- Current solution is complex

$$V(z,t) = V_o e^{-jkz} e^{jwt} = V_o e^{-j(wt-kz)}$$

- But the following solution is also valid

$$V(z,t) = V_o e^{j(wt-kz)}$$

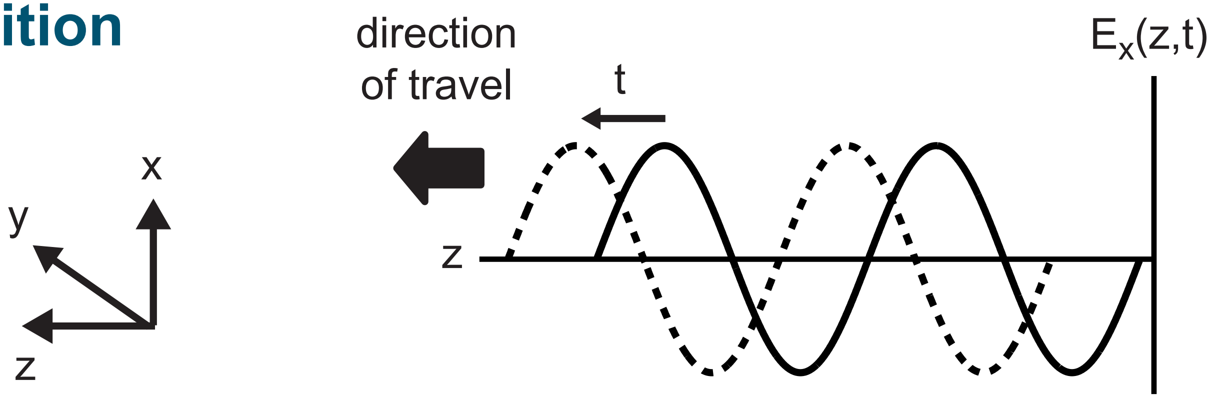
- And adding them together is also a valid solution

$$V = V_o(e^{j(wt-kz)} + e^{-j(wt-kz)})$$

$$= 2V_o \cos(wt - kz)$$

Calculating Propagation Speed

- **The resulting cosine wave is a function of time AND position**



$$V(z,t) = 2V_o\cos(wt - kz)$$

- **Consider "riding" one part of the wave**

$$-kz + wt = \text{constant (choose 0)} \Rightarrow z = \frac{wt}{k}$$

- **Velocity calculation**

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{wt}{k}\right) = \frac{w}{k} = \frac{w}{w\sqrt{LC}} = \boxed{\frac{1}{\sqrt{LC}}}$$

Integrated Circuit Values

- **Constants**

$$\epsilon = \epsilon_r \epsilon_o \quad (\epsilon_r = 3.9, 11.7, 4.4 \text{ in } SiO_2,\ Si,\ \text{FR4, respectively})$$

$$\mu = \mu_r \mu_o \quad (\mu_r = 1 \text{ for the above materials})$$

- **Impedance (geometry dependant)**

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{\mu(a/b)}{\epsilon(b/a)}} = \sqrt{\frac{\mu}{\epsilon}}\left(\frac{a}{b}\right)$$

- **Propagation speed (geometry independent)**

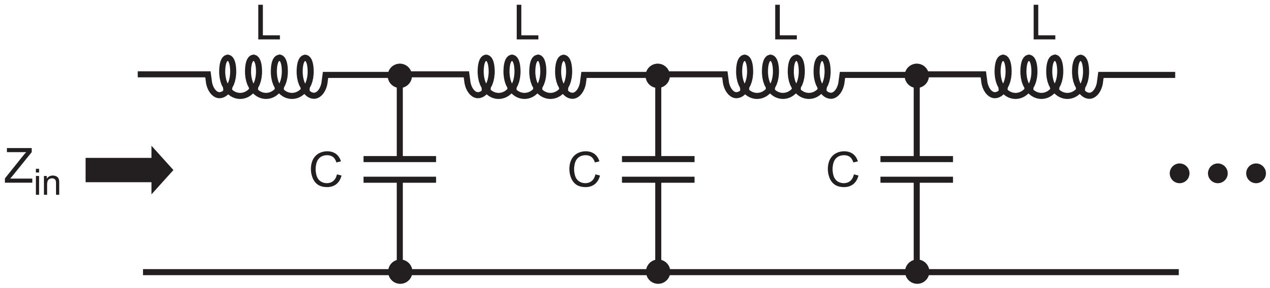
$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu(a/b)\epsilon(b/a)}} = \frac{1}{\sqrt{\mu\epsilon}} = 30\times 10^9 \text{ cm/s}$$

- **Wavelength of 30 GHz signal in silicon dioxide**

$$\lambda = \frac{T}{\sqrt{\mu\epsilon}} = \frac{1}{f\sqrt{3.9\mu_o\epsilon_o}} = 1/2 \text{ cm}$$

LC Network Analogy of Transmission Line (TEM)

- **LC network analogy**



- **Calculate input impedance**

$$Z_{in} = sL + (1/sC)||Z_{in} = sL + \frac{Z_{in}}{1 + Z_{in}sC}$$

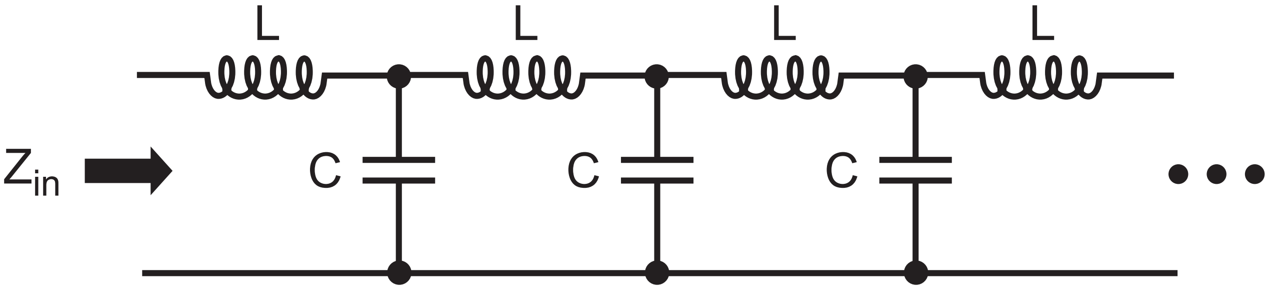
$$\Rightarrow \;\; Z_{in}^2 - sLZ_{in} - L/C = 0$$

$$\Rightarrow \;\; Z_{in} = \frac{sL}{2}\left(1 \pm \sqrt{1 + \frac{4}{s^2LC}}\right)$$

$$\text{for } |s| \ll \frac{1}{LC} \;\; \Rightarrow \;\; Z_{in} \approx \frac{sL}{2}\left(1 \pm \frac{2}{s\sqrt{LC}}\right) \approx \sqrt{\frac{L}{C}}$$

How are Lumped LC and Transmission Lines Different?

- **In transmission line, L and C values are infinitely small**
  - **It is always true that** $|s| \ll \frac{1}{LC}$

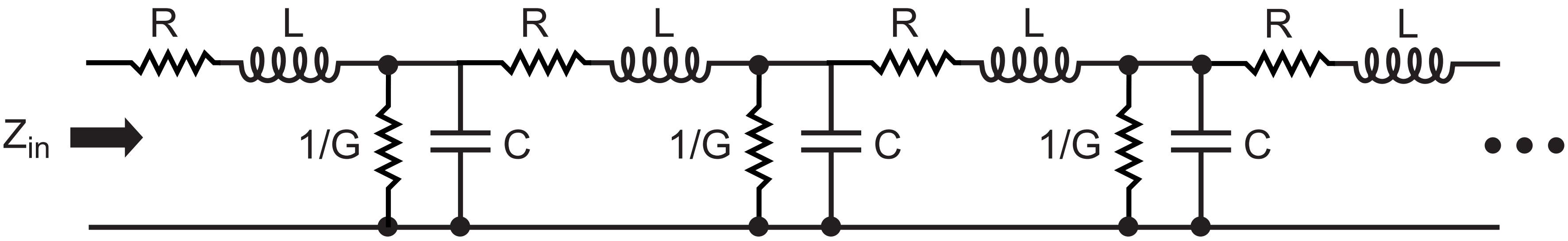


- **For lumped LC, L and C have finite values**
  - **Finite frequency range for** $|s| \ll \frac{1}{LC}$

$$Z_{in} = \frac{sL}{2}\left(1 \pm \sqrt{1 + \frac{4}{s^2LC}}\right) \Rightarrow \text{want } |s| < \frac{2}{\sqrt{LC}} \text{ for real } Z_{in}$$

Lossy Transmission Lines

- Practical transmission lines have losses in their conductor and dielectric material
  - We model such loss by including resistors in the LC model



- The presence of such losses has two effects on signals traveling through the line
  - Attenuation
  - Dispersion (i.e., bandwidth degradation)
- See Chapter 5 of Thomas Lee’s book for analysis