Philosophy of QM 24.111

Seventh lecture,

16 Feb. 2005

The statistical algorithm, continued: Multiple outcomes

We've dealt with an experiment with two outcomes.

How about an experiment ("measurement") with, say, 17 outcomes?

Answer:

(1) We need a vector space with at least 17 dimensions.

(2) We need to choose, within it, a set of 17 pairwise-orthogonal axes.

(3) We need to associate one of the outcomes with each axis.

(4) We need to find a unit vector to represent the state of the system being "measured".

Then we can apply the statistical algorithm in the same way that we did, in the case of spin measurements.

Axes and orthonormal bases, part 1

First let's talk about bases for a vector space. Let's work with R^2.

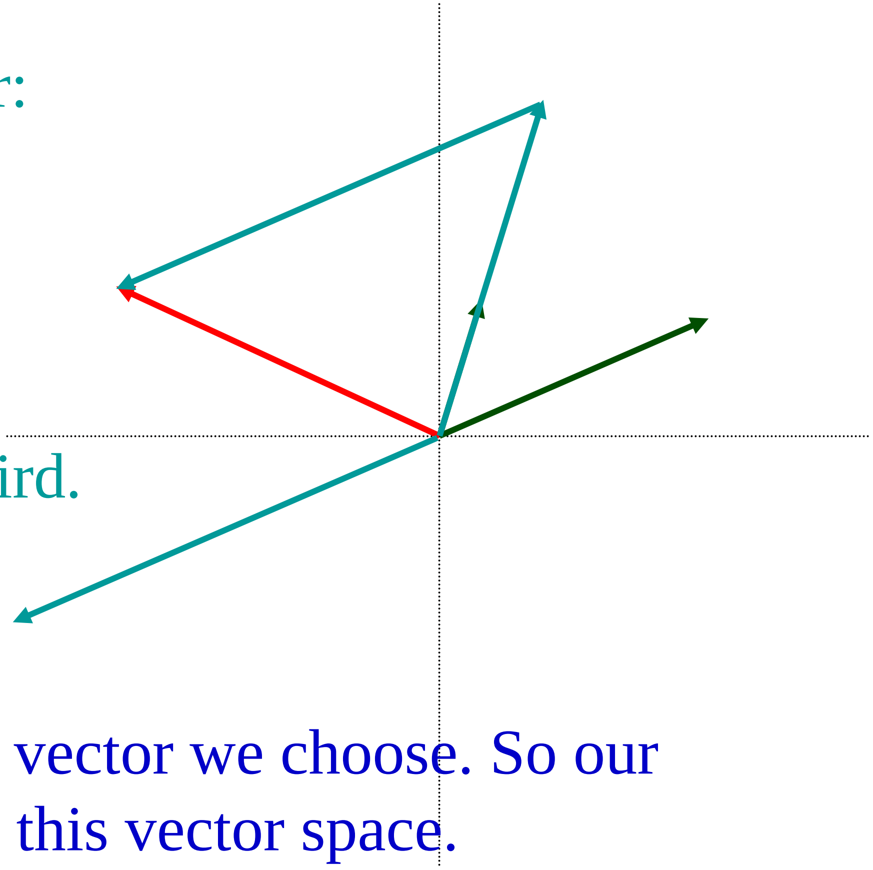
Pick any two non-parallel vectors:

Now pick some arbitrary vector:

We can scale the first vector,

flip and scale the second,

and add the results to get the third.

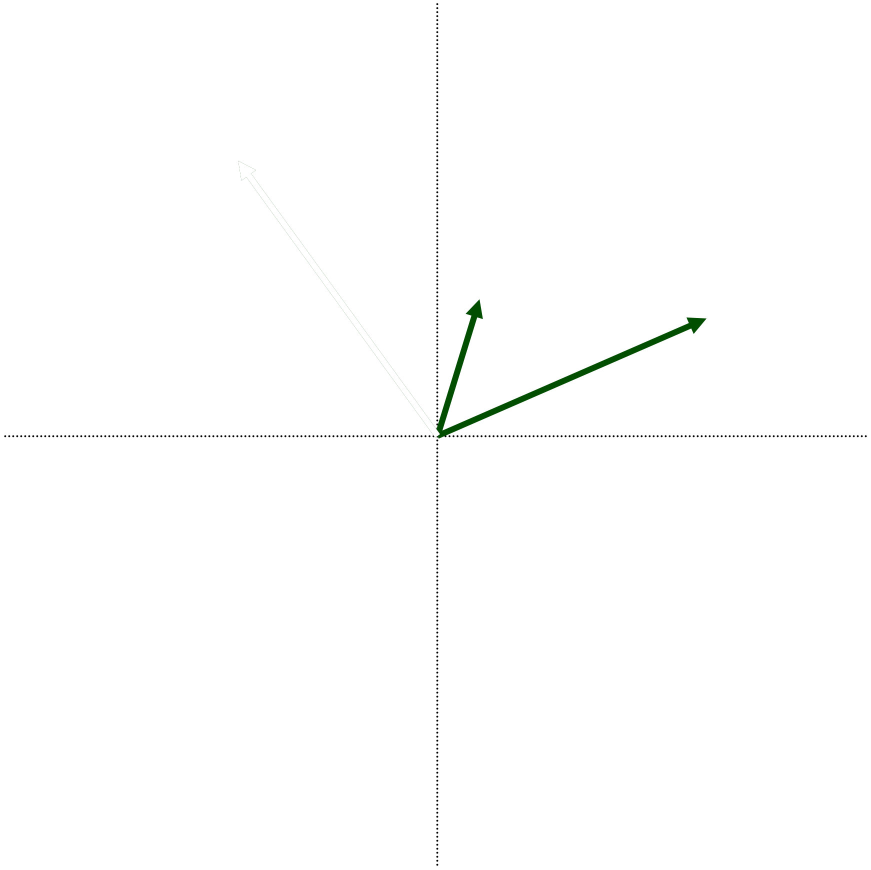


This is true no matter which red vector we choose. So our green vectors form a **BASIS** for this vector space.

Axes and orthonormal bases, part 2

It is very convenient to work with bases that

(1) contain no redundant elements;

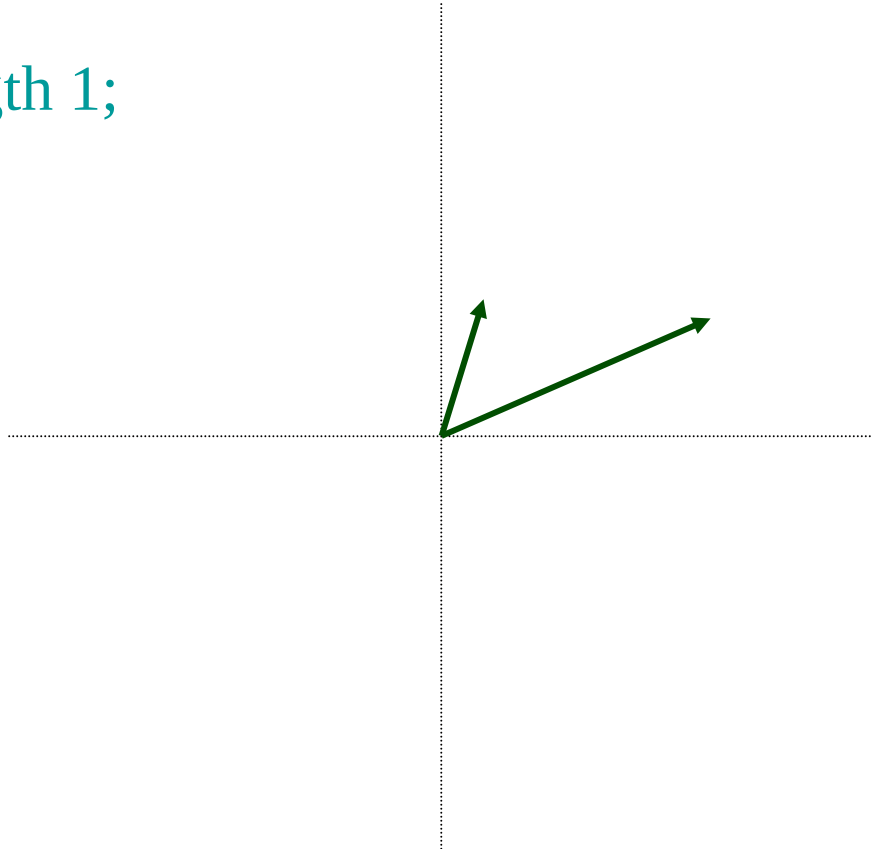


Axes and orthonormal bases, part 2

It is very convenient to work with bases that

(1) contain no redundant elements;

(2) contain only vectors of length 1;



Axes and orthonormal bases, part 2

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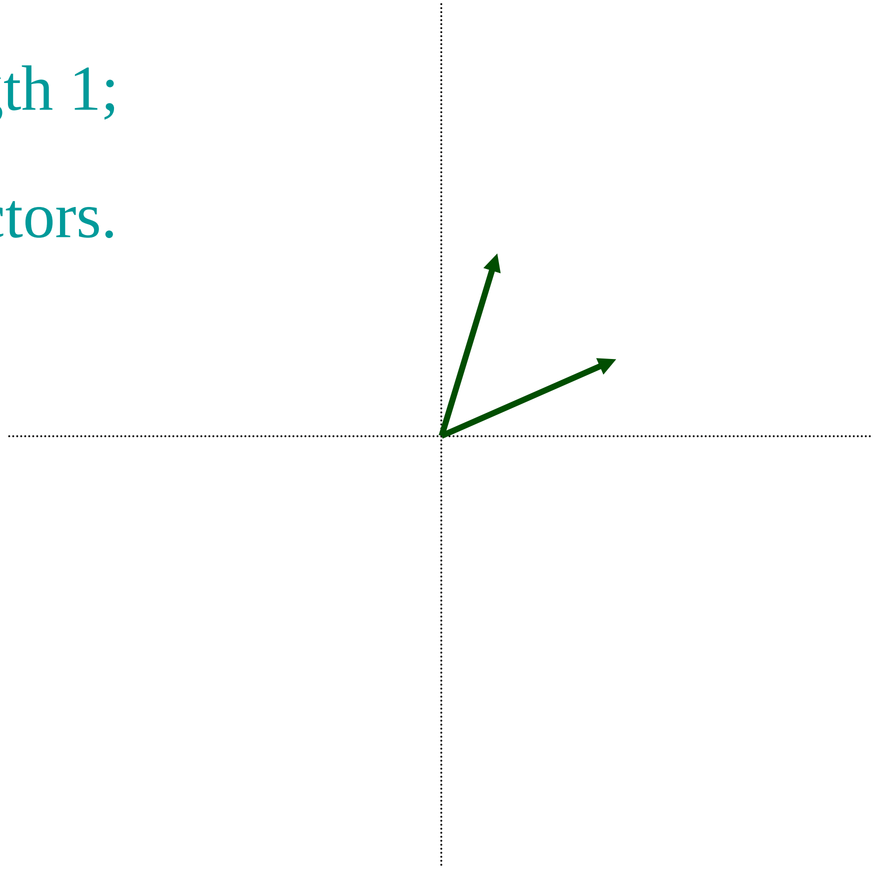

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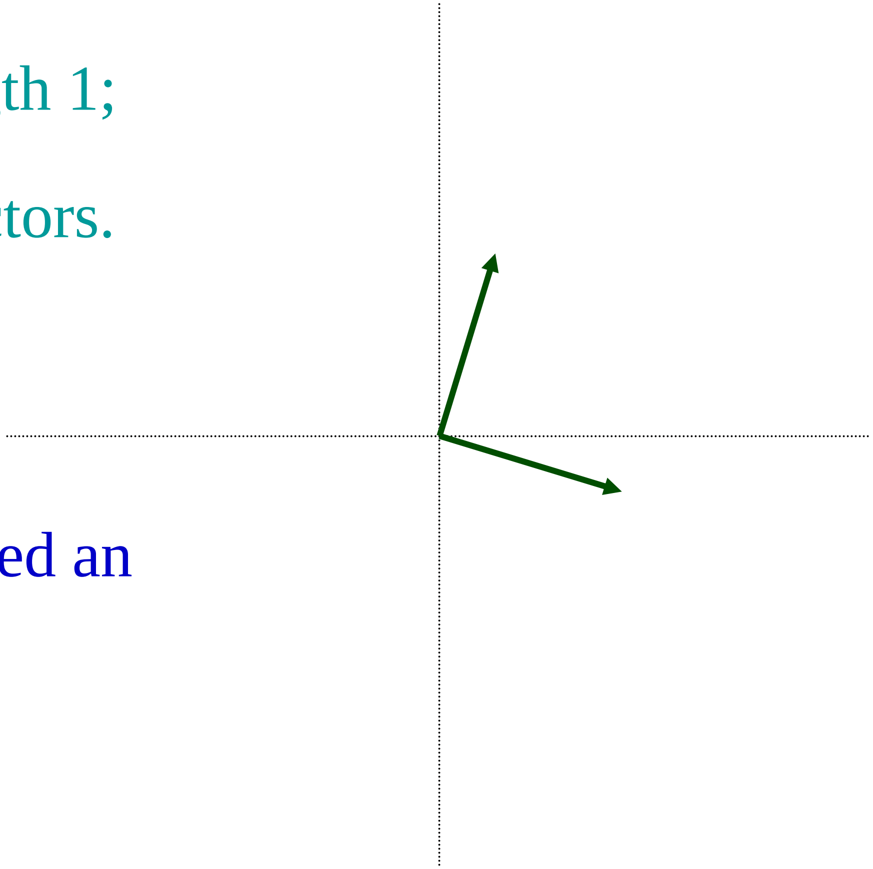


A basis with these features is called an **orthonormal basis**.

Projection and inner product, part 1

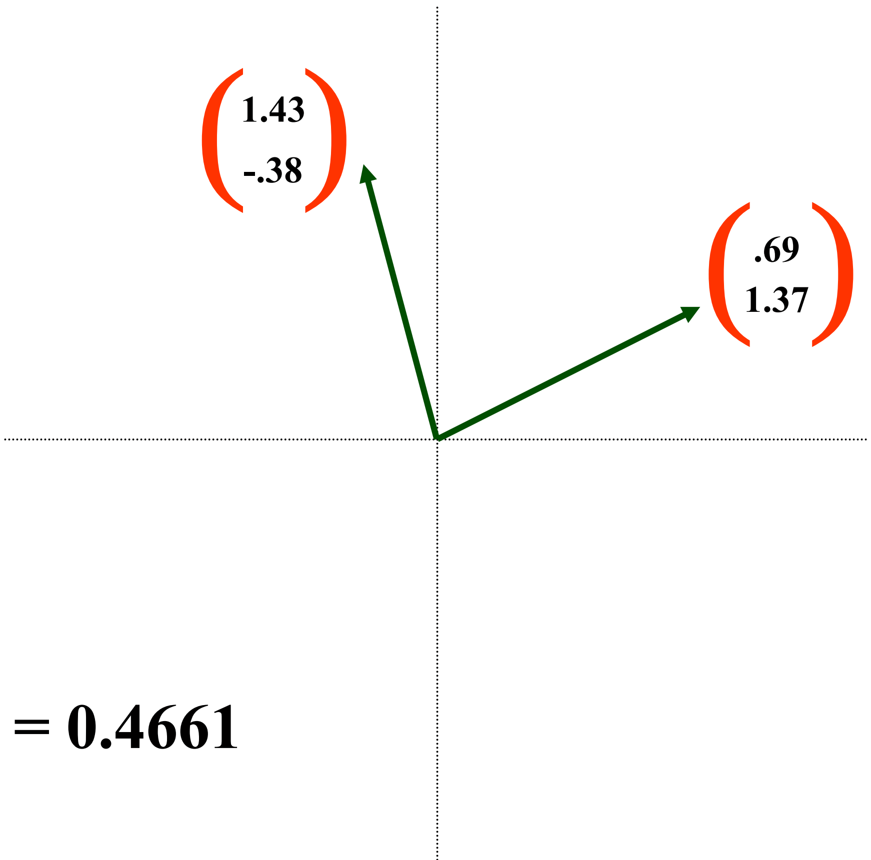
Vectors can—and should!—be thought of as pairs (or triples, or quadruples, etc.) of numbers (real or complex):

Given two such vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix}$$

their inner product is defined as

$$\begin{pmatrix} a \\ b \end{pmatrix} \bullet \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd.$$



In our example, $\begin{pmatrix} 1.43 \\ -.38 \end{pmatrix} \bullet \begin{pmatrix} .69 \\ 1.37 \end{pmatrix} = 0.4661$

Projection and inner product, part 2

> Observe that since N has length 1, the length of <**V** | **N**>**N** is just <**V** | **N**>.

How to project a vector V onto an axis:

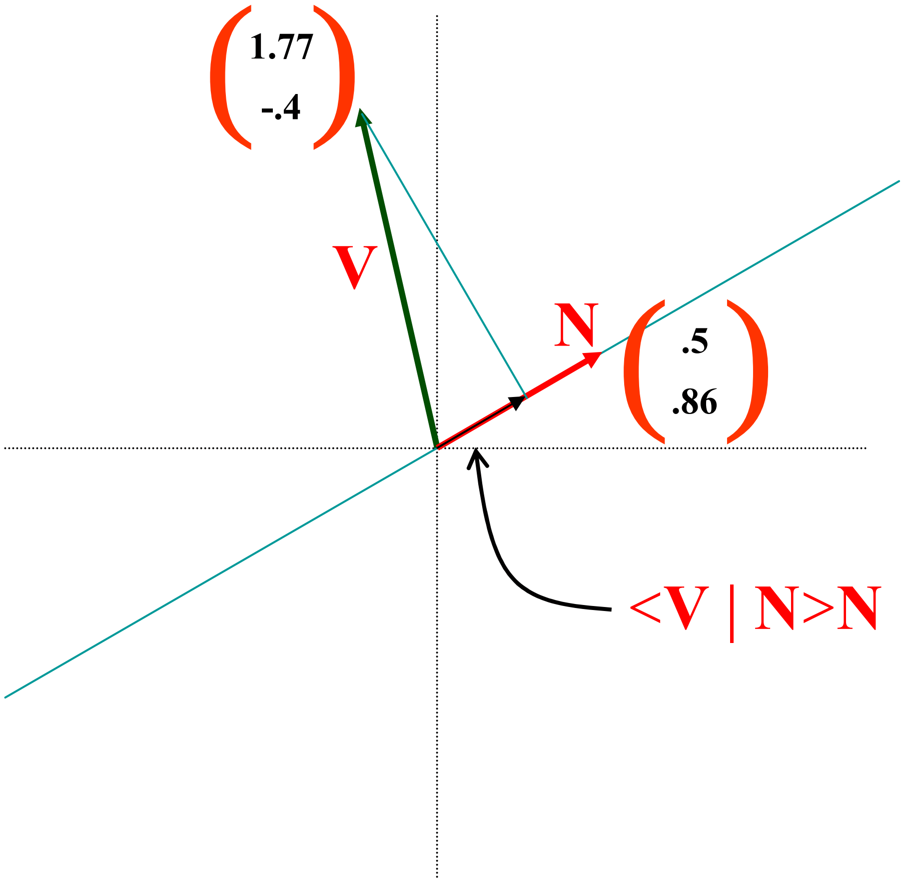
1. Pick a length-1 vector **N** lying on the axis (any will do).

2. Calculate <**V** | **N**>:

$$\begin{pmatrix} 1.77 \\ -.4 \end{pmatrix} \bullet \begin{pmatrix} .5 \\ .86 \end{pmatrix} = 0.541$$

3. Multiply **N** by the result:

$$.541 \begin{pmatrix} .5 \\ .86 \end{pmatrix} = \begin{pmatrix} .2705 \\ .46526 \end{pmatrix}$$



Restating the statistical algorithm

This means that in stating the statistical algorithm, we can replace talk of orthogonal axes and projection with talk of orthonormal bases and inner products:

Suppose system S is in a state represented by the unit vector **V**.

Suppose experiment E is performed on S, where E is represented by the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$, with vector $\mathbf{e}_i$ corresponding to outcome i.

To calculate **Prob(outcome i)**, calculate $<\mathbf{V} \mid \mathbf{e}_i>^2$.

Redundancy in the state-vector

Observe that if **W** = **-V**, then for any vector **e**,

$$\langle \mathbf{W} \mid \mathbf{e} \rangle^2 = \langle \mathbf{V} \mid \mathbf{e} \rangle^2.$$

That means that as far as the statistical algorithm is concerned, **W** and **V** represent exactly the same physical state.

(The same holds for **W = cV**, where **c** is a complex number with absolute value 1.)

State-vectors and experiments

(1) The physical state of any system is represented by a vector in some vector space (usually an infinite-dimensional vector space; note that this will be a different vector space for each different system).

(2) If Φ is a vector representing one possible physical state of some system, and Ψ is another vector representing another possible physical state of that system, then any arbitrary linear combination $a\Phi + b\Psi$ also represents a possible physical state of the system. This is called the principle of superposition.

(3) Any experiment that can be performed on a system is represented by an orthonormal basis in the vector space for that system. Each of basis element can be thought of as "labeled" with one of the possible outcomes of the experiment.

Position "measurements"

There is a particle somewhere in the room. We can look for it

here,

or here,

or here,

…etc. It turns out that we need an infinite-dimensional vector space if we are to represent such position "measurements". It also turns out that the very same vector space can be used to represent "measurements" of momentum, kinetic energy, and a number of other "observables". We will come back to this later in the course.

Dirac notation

Assume: Every orthonormal basis represents some possible experiment.

Then: for every state-vector **V**, there is some experiment **E** and outcome α such that if the system has state **V**, then **E** is certain to produce α.

This fact gives us a handy way to write down the vector for a system: pick some such experiment and outcome, and describe the state-vector by reference to them.

Ex: a spin-1/2 particle certain to go up through 0° is in state **| up, 0° >**. A particle "located" in region R—in the sense that a measurement of its position will, with certainty, reveal it to be in R—is in state **| in R >**.

Relations among spin states

This way of writing down state-vectors leaves it opaque what their mathematical relations are—what linear combinations of them result in which others of them. Here are some useful relations; they are pretty much the only ones we will need:

| up, 90° > = 1/√2 | up, 0° > + 1/√2 | down, 0° >

| down, 90° > = 1/√2 | up, 0° > - 1/√2 | down, 0° >

Spin and position aspects of states

For the experiments we will analyze, it is very convenient to keep separate track of those aspects of a particle's state that have to do with its "spin"—i.e., that determine the probabilities that it will get deflected up or down through any given magnet—from those having to do with its position (and momentum, although we won't care so much about that). To do this, we will represent its state, in the simplest case, by a "product" of vectors. For example, if a particle is located in region of space R, and is certain to go up through a 0°-magnet, we will write its state as **| up, 0°> | in R >**.

“Entangled” states

Not all states of spin-1/2 particles can be written in this way. Suppose that R and S are two distinct regions of space. Then

| up, 0°> | in R >

is one possible state-vector for our particle; so is

| down, 0°> | in S >.

The superposition principle tells us that any linear combination of two state-vectors is itself a state-vector. So

1/√2 | up, 0°> | in R > + 1/√2 | down, 0°> | in S >

is also a state-vector. **You cannot rewrite this as a simple “product” of a spin part and a position part.** Spin and position are “entangled”.

Tensor products: some warnings

Suppose we're given a vector

1/√2 | up, 0°> | in R > + 1/√2 | down, 0°> | in S >

Can we rewrite this as

1/√2 (| up, 0°> + | down, 0°>)(| in R > + | in S >)?

NO!!! These tensor products work like multiplication: **xy + zw ≠ (x+z)(y+w).** Note, however, that they do distribute, so that

1/√2 | up, 0°> | in R > + 1/√2 | down, 0°> | in R >

can be rewritten as

1/√2 (| up, 0°> + | down, 0°>) | in R > = | up, 90°> | in R >

Schrodinger's Equation

Suppose that our system starts out in state Φ, and changes, over some time (5 minutes, say), into state Φ'. Of course, it could have started out in any of a multitude of *different* states. So suppose it starts out in state Ψ, and changes over the 5 minute interval into state Ψ'. We can schematically represent these two possible "trajectories" thus:

Φ → Φ'

Ψ → Ψ'.

Since Φ and Ψ are possible states of the system, so is their arbitrary linear combination aΦ + bΨ. What Schrodinger's Equation tells us is that given that Φ and Ψ would change in the ways just indicated, their linear combination must change in the following way:

aΦ + bΨ → aΦ' + bΨ'.

That fact turns out to make it surprisingly easy to analyze what look like complex experiments. **Note that SE holds only if no "measurement" is taking place.**

The "collapse of the wave-function"

If a "measurement" is taking place—say, of some system S—then an entirely different story gets told about how the state of the system changes: during the measurement, the system S must "jump" (sometimes we say "collapse") into a state that is certain to produce the observed result of the measurement. For example, suppose we have a particle in the state

$$1/\sqrt{2}\ |\ \text{up},\ 0^\circ > |\ \text{in R} > +\ 1/\sqrt{2}\ |\ \text{down},\ 0^\circ > |\ \text{in S} >,$$

and we look for it by placing a detector in region R. If we find it there (if our detector triggers, that is), then its state instantly changes to

$$|\ \text{up},\ 0^\circ > |\ \text{in R} >.$$

On the other hand, if we don't find it there (our detector does not trigger), then its state instantly changes to

$$|\ \text{down},\ 0^\circ > |\ \text{in S} >.$$