24.118 – Paradox and Infinity
Problem Set 7: The Axiom of Choice

You will be graded both on the basis of whether your answers are correct and on the basis of whether they are properly justified. There is no word limit. (Three exceptions: problems 2b, 3c and 3d will be assessed on the basis on the *reasons* you give in support of your answers, rather than the answers themselves. But you must give your answers in less than 150 words each.)

1. *The Axiom of Choice*

   The Axiom of Choice is an axiom of set theory which has weird consequences, and is sometimes regarded as controversial. It is often formulated as follows:

   > Axiom of Choice
   > Let α be a set, each of whose members is a non-empty set. Then there is a function f_α such that, for any $\beta \in \alpha$, $f_\alpha(\beta) \in \beta$. (In other words, f_α is a 'choice function' for α.)

   A different, but equivalent, formulation of the Axiom of Choice is the following:

   > The Well-Ordering Theorem
   > Every set can be well ordered.[1]

   Problem: Prove the Axiom of Choice using the Well-Ordering Theorem. (5 points)

2. *The Hat Problem*

   An infinite group of people P_1, P_2, P_3, ... (one for each natural number) are standing in line. P_1 is behind P_2, P_3, ...; P_2 is behind P_3, P_4, ...; and so forth. At time t_0 everyone is asked to close their eyes. Each person is approached by an 'assistant'. The assistant flips a coin. If the coin lands Heads, the assistant places a blue hat on the relevant person's head; if the coin lands Tails, the assistant places a red hat on the relevant person's.

   Once everyone has a hat, people are allowed to open their eyes. Nobody can see the color of their own hat, or of the finitely many hats behind them. But everyone can

[1] Remember our discussion of well-orderings from last week. Intuitively, a set α is well-ordered by a relation $\leq$ if every non-empty subset of α has a $\leq$-smallest element. (So, for example, the natural numbers are well-ordered by the standard less-than-or-equal relation, but the negative integers are not.)

Formally, α can be *well-ordered* if and only if there is a relation $\leq$ which is a total order on α (i.e. for any $a, b, c \in \alpha$: (1) either $a \leq b$ or $b \leq a$, (2) if $a \leq b$ and $b \leq a$, then $a = b$, and (3) if $a \leq b$ and $b \leq c$, then $a \leq c$) and which is such that any non-empty subset of α has a $\leq$-smallest element.

see the colors of the infinitely many hats ahead of them. (The coin tosses are all independent of each other, so the colors of other people's hats gives you no information about the color of your own hat.)

At a later time, t_1, everyone is asked to guess the color of their hat. If only finitely many people guess wrong, everyone's life is spared. But if infinitely many people guess wrong, everyone is killed.

Problems:

(a) Find a strategy that P_1, P_2, P_3, ... can agree upon ahead of time which would guarantee that at most finitely many people get the wrong answer. (10 points)

Hint 1: Consider the space of all possible sequences of hat-distributions, and partition it into equivalence classes such that two sequences are in the same equivalence class just in case they disagree in at most finitely many places. So, for example, the sequences

R, R, R, R, ...
R, B, R, B, ...

are in different equivalence classes because they disagree in all even positions (of which there are infinitely many). On the other hand, the sequences

R, R, R, R, ...
B, R, R, R, ...

are in the same equivalence class because they only disagree in the first position.

Hint 2: Use the Axiom of Choice.

(b) Suppose you are one of the people playing the game. Are you more likely to guess the color of your hat right if you follow the proposed strategy than if you guess at random? (5 points)

3. *The Square of Evil*

The Square of Evil is the unit square [0,1]×[0,1], where certain points have been colored white and others have been colored black.

In deciding which points to color white and which ones to color black we will assume both The Well-Ordering Theorem and The Continuum Hypothesis. (The Continuum Hypothesis is the assertion that there is no size of infinity between the size of the natural numbers and the size of the real numbers.) These two assumptions together entail the existence of an 'evil' well-ordering $\leq^e$ on [0,1] with the following property: for each $x \in [0, 1]$, there are at most countably many $y \in [0, 1]$ such that $y \leq^e x$. (A set is *countable* if it can be put in one-one correspondence with a subset of the natural numbers.)

Here is how to use the evil well-ordering to color the unit square [0,1]×[0,1]. For each point $\langle x, y\rangle \in [0,1] \times [0,1]$, color $\langle x, y\rangle$ white just in case $x \leq^e y$; otherwise color it black. (This construction is due to the Polish mathematician Wacław Sierpiński.)

Problems:

(a) Let $\langle x_0, y_0\rangle$ be a point on the Square of Evil. How many white points are there in the column $\{\langle x_0, z\rangle : z \in [0,1]\}$? How many white points are there in the row $\{\langle z, y_0\rangle : z \in [0,1]\}$? (5 points)

(b) Describe a well-defined procedure for picking a point at random from the Square of Evil. [*Hint:* use an infinite sequence of coin-tosses.] (3 points)

(c) Pick a point at random from the Square of Evil. Call it $\langle x_0, y_0\rangle$.

  i. Suppose you learn that most points on the column $\{\langle x_0, z\rangle : z \in [0,1]\}$ are white. If you are forced to bet on the color $\langle x_0, y_0\rangle$, should you bet Black or bet White? (1 point)

  ii. Suppose you learn that most points on the row $\{\langle z, y_0\rangle : z \in [0,1]\}$ are black. If you are forced to bet on the color $\langle x_0, y_0\rangle$, should you bet Black or bet White? (1 point)

(d) Pick a point at random from the Square of Evil. If you are forced to bet on the color of the point, should you bet Black or bet White? (3 points)

(e) *Extra Credit:* Earlier I claimed that the Well-Ordering Theorem and the Continuum Hypothesis entail that there is a well-ordering $\leq^e$ on [0,1] with the property that for any $x \in [0,1]$, there are at most natural-number many $y \in [0,1]$ such that $y \leq^e x$. Show that this is true. (5 points)

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