

3.185 Problem Set 8

Drag Force, Advanced Heat and Mass Transfer

Solutions

1. Settling of magnesia particles in water

(a) For a sphere of diameter d in a fluid, the net gravity/buoyancy force is given by:

$$F_w - F_b = \frac{1}{6}\pi d^3 g(\rho_s - \rho)$$

where ρ_s is the density of the sphere, and ρ that of the fluid. When the sphere is rising or sinking at its terminal velocity V , that force is exactly balanced by the drag force, given by:

$$|F_w - F_b| = F_d = fKA = f \cdot \frac{1}{2}\rho V^2 \cdot \frac{1}{4}\pi d^2$$

where the friction factor f in Stokes flow is:

$$f = \frac{24}{\text{Re}} = \frac{24\mu}{\rho V d}$$

resulting in a drag force of:

$$F_d = 3\pi\mu V d$$

We set this equal to $|F_w - F_b|$ and solve for V :

$$\frac{1}{6}\pi d^3 g|\rho_s - \rho| = |F_w - F_b| = F_d = 3\pi\mu V d$$

$$V = \frac{d^2 g|\rho_s - \rho|}{18\mu}$$

(b) Set $V = 10\text{cm}/1\text{min} = \frac{1}{600}\frac{\text{m}}{\text{s}}$ and solve for d :

$$d = \sqrt{\frac{18\mu V}{g|\rho_s - \rho|}} = \sqrt{\frac{18 \cdot 10^{-3} \frac{\text{kg}}{\text{m}\cdot\text{s}} \cdot \frac{1}{600} \frac{\text{m}}{\text{s}}}{9.8 \frac{\text{m}}{\text{s}^2} |3850 - 1000| \frac{\text{kg}}{\text{m}^3}}}$$

$$d \simeq 33\mu\text{m}$$

All of the particles larger than $33\mu\text{m}$ across will sink faster than that, and reach the bottom within one minute.

(c) The Reynolds number is:

$$\text{Re} = \frac{\rho V d}{\mu} = \frac{1000 \frac{\text{kg}}{\text{m}^3} \cdot \frac{1}{600} \frac{\text{m}}{\text{s}} \cdot 3.3 \times 10^{-5} \text{m}}{10^{-3} \frac{\text{kg}}{\text{m}\cdot\text{s}}} = 0.054$$

This is below 0.1, so we're safely in the Stokes law régime.

2. Drag force on a flat plate

- (a) Drag force is proportional to the width and to the square root of the length, so it is lower for the longer length and narrower width, *i.e.* 1m edges parallel to the wind.
- (b) First we need to know whether flow is laminar or otherwise, which comes from the Reynolds number:

$$\text{Re}_L = \frac{\rho U_\infty L}{\mu} = \frac{1.9 \frac{\text{kg}}{\text{m}^3} \cdot 0.01 \frac{\text{m}}{\text{s}} \cdot 1\text{m}}{10^{-5} \frac{\text{N}\cdot\text{s}}{\text{m}^2}} = 1.9 \times 10^3$$

Since this is below 10^5 , laminar sounds good. Also, the ratio of boundary layer thickness to length $\frac{\delta}{L}$ is about 0.11, so the $x \gg \delta$ assumption holds and the boundary layer analysis with it. Whether 11 cm is much less than 25 cm (so we can neglect edge effects) is possibly arguable, but we'll accept it for an estimate.

Now we have the dimensionless correlation for drag force due to laminar flow past a flat plate:

$$f_L = \frac{1.328}{\sqrt{\text{Re}_L}} = 0.0305$$

$$F_d = f K A = f \times \frac{1}{2} \rho U_\infty^2 \times L W$$

$$F_d = 0.0304 \times \frac{1}{2} \cdot 1.9 \frac{\text{kg}}{\text{m}^3} \cdot \left(0.01 \frac{\text{m}}{\text{s}}\right)^2 \times 1\text{m} \cdot 0.25\text{m} = 7.24 \times 10^{-7}\text{N}$$

This is the force on one side of the plate; the total force is twice this, about $1.5\mu\text{N}$. That's pretty small!

- (c) The wind velocity is 1000 times larger than before, so Re_L is 1.9×10^6 , and flow is transitional over much of the plate. The ratio $\frac{\delta}{L}$ (using the turbulent flow correlation) is now about 0.021, so we have a thinner boundary layer than before. (If flow were still laminar, it would be about 30 times thinner than in part 2b due to the larger velocity, so the turbulence effectively makes it about 10 times thicker than a laminar boundary layer would be.)

There are two different correlations for the drag force. First the one from P&G p. 83:

$$f = \frac{0.455}{(\log \text{Re}_L)^{2.58}} = \frac{0.455}{6.28^{2.58}} = 4.0 \times 10^{-3}$$

And from BSL p. 203:

$$f = \frac{0.146}{\text{Re}_L^{0.2}} = 8.1 \times 10^{-3}$$

So the two friction factors are a factor of two apart!

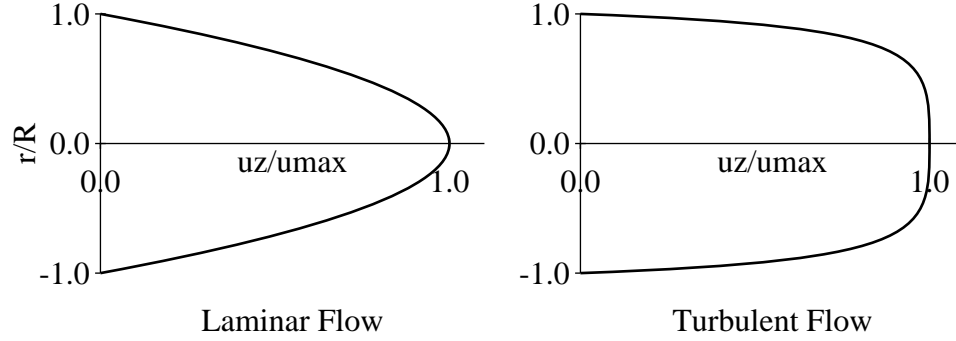
The force is calculated using the same $F_d = f K A$ equation as in part 2b, and we get

$$F_d = 4.0 \times 10^{-3} \times \frac{1}{2} \cdot 1.9 \frac{\text{kg}}{\text{m}^3} \cdot \left(10 \frac{\text{m}}{\text{s}}\right)^2 \times 1\text{m} \cdot 0.25\text{m} = 0.095\text{N}$$

(or about twice this using the BSL friction factor correlation); again double this for both sides of the plate. This is still pretty small.

3. Turbulence and Mixing in a Tube

- (a) The laminar velocity profile is parabolic, and the turbulent profile looks somewhat like pseudo-plastic flow, since there's less mixing, and lower turbulent viscosity, and steeper velocity gradient near the tube walls away from the center. Your sketches should have looked something like:



- (b) Start by calculating the average velocity, to get the Reynolds number and friction factor:

$$\bar{u} = \frac{Q}{A_{xs}} = \frac{0.001 \frac{\text{m}^3}{\text{s}}}{\pi \cdot \left(\frac{0.01\text{m}}{2}\right)^2} = 12.7 \frac{\text{m}}{\text{s}}$$

$$\text{Re} = \frac{\bar{u}d}{\nu} = \frac{12.7 \frac{\text{m}}{\text{s}} \cdot 0.01\text{m}}{10^{-6} \frac{\text{m}^2}{\text{s}}} = 1.27 \times 10^5$$

At this Reynolds number, and the dimensionless roughness $\epsilon/d = 10^{-5}\text{m}/10^{-2}\text{m} = 10^{-3}$, the graph provided gives a friction factor of approximately 5.5×10^{-3} .

Next we set the drag force magnitude equal to the pressure force, since they're equal and opposite, and solve for ΔP :

$$F_d = fKA_{tube} = \Delta P \cdot A_{xs}$$

$$f \cdot \frac{1}{2} \rho \bar{u}^2 \cdot 2\pi RL = \Delta P \cdot \pi R^2$$

$$\Delta P = \frac{L}{R} f \rho \bar{u}^2 = \frac{30\text{m}}{0.005\text{m}} \cdot 5.5 \times 10^{-3} \cdot 1000 \frac{\text{kg}}{\text{m}^3} \cdot \left(12.7 \frac{\text{m}}{\text{s}}\right)^2 = 5.35 \times 10^6 \frac{\text{N}}{\text{m}^2}$$

Q.E.D.

- (c) The Hagen-Poiseuille equation states:

$$Q = \frac{\pi R^4 \Delta P}{8\mu L}$$

This is used for laminar flow, so in this turbulent case, it gives a good estimate of the average turbulent viscosity in this tube. Just substitute μ_t for μ and solve for μ_t :

$$\mu_t = \frac{\pi R^4 \Delta P}{8QL} = \frac{\pi \cdot (0.005\text{m})^4 \cdot 5.35 \times 10^6 \frac{\text{N}}{\text{m}^2}}{8 \cdot 0.001 \frac{\text{m}^3}{\text{s}} \cdot 30\text{m}} = 0.0438 \frac{\text{kg}}{\text{m} \cdot \text{s}}$$

So the turbulent viscosity is more than forty times the molecular viscosity of $10^{-3} \frac{\text{kg}}{\text{m} \cdot \text{s}}$.

- (d) Use the turbulent Prandtl number of one to estimate the turbulent diffusivity:

$$\text{Pr}_t = \frac{\nu_t}{D_t} = \frac{\mu_t}{\rho D_t} \simeq 1$$

$$D_t \simeq \frac{\mu_t}{\rho} = \frac{0.0438 \frac{\text{kg}}{\text{m} \cdot \text{s}}}{1000 \frac{\text{kg}}{\text{m}^3}} = 4.38 \times 10^{-5} \frac{\text{m}^2}{\text{s}}$$

This mixes across the tube in approximately the diffusion timescale of the diameter (or radius is okay):

$$t_{ss} = \frac{d^2}{D_t} = \frac{(0.01\text{m})^2}{4.38 \times 10^{-5} \frac{\text{m}^2}{\text{s}}} = 2.28\text{seconds}$$

This is the time required to mix (but not homogenize) the substance across the width of the tube, and is nearly the time required to travel the length of the tube!

- (e) The power is the flow rate times pressure drop: $\text{power} = Q\Delta P$ and the energy dissipation rate per unit volume is simply the power dissipated divided by the volume: $\epsilon = \text{power}/V$. If we assume that all of the power goes into turbulent dissipation, this gives us:

$$\epsilon = \frac{Q\Delta P}{\pi R^2 L} = \frac{0.001 \frac{\text{m}^3}{\text{s}} \cdot 5.35 \times 10^6 \frac{\text{N}}{\text{m}^2}}{\pi \cdot (0.005\text{m})^2 \cdot 30\text{m}} = 2.27 \times 10^6 \frac{\text{N}}{\text{m}^2 \cdot \text{s}}$$

Since one newton-meter is a joule, and a joule per second is a watt, this is $2.27 \times 10^6 \frac{\text{W}}{\text{m}^3}$.

- (f) The expression for turbulent microscale was given on the equation sheet:

$$\ell \simeq \sqrt[4]{\frac{\mu^3}{\rho^2 \epsilon}}$$

The dissipation rate ϵ (not the surface roughness ϵ) comes from part 3e, and the viscosity used is the *molecular* viscosity (not the turbulent viscosity) because molecular viscosity is turning mechanical power into heat energy in these small eddies, turbulent viscosity mixes the fluid at much larger lengthscales. So we have:

$$\ell \simeq \sqrt[4]{\frac{\left(10^{-3} \frac{\text{kg}}{\text{m}\cdot\text{s}}\right)^3}{\left(1000 \frac{\text{kg}}{\text{m}^3}\right)^2 \cdot 2.27 \times 10^6 \frac{\text{W}}{\text{m}^3}}} = \sqrt[4]{4.40 \times 10^{-22} \text{m}^4} = 4.6 \times 10^{-6} \text{m}$$

About five microns.

- (g) Turbulence mixes the substance to the lengthscales in part 3f, then molecular diffusion has to homogenize it the rest of the way. The timescale for this second part is simply the steady-state diffusion time for the small eddies:

$$t_{\text{homog}} \simeq t_{ss} \simeq \frac{\ell^2}{D} = \frac{(4.6 \times 10^{-6} \text{m})^2}{10^{-6} \frac{\text{cm}^2}{\text{s}} \cdot \left(\frac{1\text{m}}{100\text{cm}}\right)^2} = 0.210 \text{seconds}$$

So in this case, the substance is mixed across the tube in about two seconds, and diffuses on the eddy lengthscales it about two tenths of a second. It should be completely homogenized within about 2.5 seconds, which for fluid travelling 30 m at 12.7 m/s is about the amount of time the fluid spends in the tube.

4. Chemical Vapor Deposition

- (a) The mass transfer Prandtl number is given by:

$$\text{Pr} = \frac{\mu}{\rho D}; \quad \rho = \frac{MP}{RT},$$

where M is the molar mass. Since $1\text{atm}=101300 \text{ Pa}$, this pressure is 10130 Pa ; Pascals are equivalent to Joules per cubic meter. This therefore evaluates to:

$$\rho = \frac{0.040 \frac{\text{kg}}{\text{mol}} \cdot 10130 \frac{\text{J}}{\text{m}^3}}{8.314 \frac{\text{J}}{\text{K}\cdot\text{mol}} \cdot 500\text{K}} = 0.097 \frac{\text{kg}}{\text{m}^3}; \quad \text{Pr} = \frac{3 \times 10^{-5} \frac{\text{N}\cdot\text{s}}{\text{m}^2}}{0.097 \frac{\text{kg}}{\text{m}^3} \cdot 2 \times 10^{-4} \frac{\text{m}^2}{\text{s}}} = 1.54.$$

This is not far from one, which is not atypical for a gas (yes, that was a double-negative).

- (b) With a flow rate of $0.2 \frac{\text{m}^3}{\text{s}}$, and box width and height of $2\text{m} \times 0.5\text{m}$, the average velocity is 0.2m/s . The Reynolds number is then:

$$\text{Re} = \frac{\rho u_{av} H}{\mu} = \frac{0.097 \frac{\text{kg}}{\text{m}^3} \cdot 0.2 \frac{\text{m}}{\text{s}} \cdot 0.5\text{m}}{3 \times 10^{-5} \frac{\text{N}\cdot\text{s}}{\text{m}^2}} = 325.$$

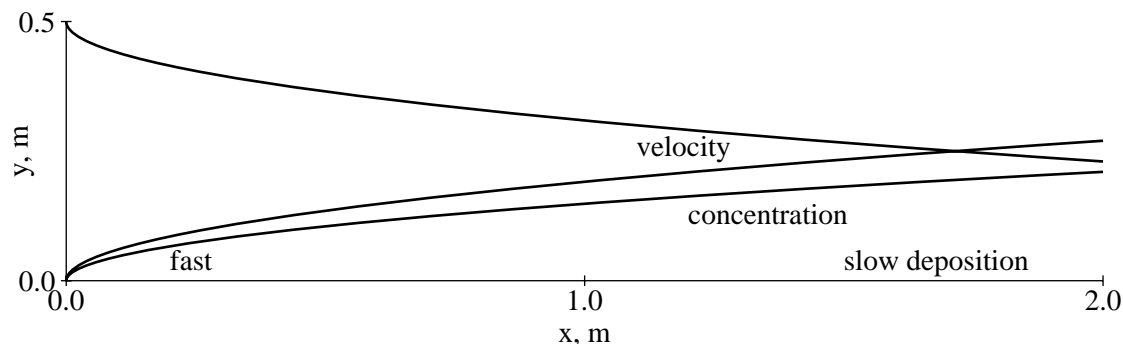
Looks like flow will remain laminar.

- (c) The velocity boundary layer thickness will grow as:

$$\delta_u = 5.0 \sqrt{\frac{\nu x}{U_\infty}},$$

for which we can use u_{av} as U_∞ ; this gives a maximum thickness of 0.27m at $x = 2m$. The velocity boundary layers will just meet at the end of the reactor.

With a Prandtl number just above one, the concentration boundary layer will be somewhat smaller than the velocity boundary layer, but our expression for relative boundary layer thickness won't quite be right. So we can estimate that the velocity and concentration boundary layers will look something like:



- (d) Though the concentration boundary layer thickness correlation is not quite right, the high-Pr Nusselt number correlation is accurate down to a Prandtl number of 0.5, so we can use that:

$$\text{Nu}_x = 0.332 \text{Re}_x^{1/2} \text{Pr}^{1/3},$$

along with expressions for the local mass transfer coefficient h_{Dx} and diffusive flux:

$$\text{Nu}_x = \frac{h_{Dx} x}{D_{fl}} \Rightarrow h_{Dx} = \frac{\text{Nu}_x D_{fl}}{x}; \quad J_y|_{y=0} = h_{Dx} (C_\infty - C_s).$$

The molar density of the gas mixture is P/RT , so with 1 mol% silane, we can use one percent of the overall pressure as the silane partial pressure and set $C_\infty = P_{\text{SiH}_4}/RT$. The problem states that we can estimate $C_s = 0$. The results are summarized as follows:

Distance x	Re_x	Nu_x	h_{Dx}	$J_y _{y=0}$
0.10m	65	3.08	$6.17 \times 10^{-3} \text{m/s}$	$1.5 \times 10^{-4} \text{mol/m}^2 \text{sec}$
0.30m	195	5.34	$3.56 \times 10^{-3} \text{m/s}$	$8.68 \times 10^{-5} \text{mol/m}^2 \text{sec}$

Note that this close to the entrance, we're really pushing the envelope of validity of this solution (since $\delta_C \ll x$ is not so valid). Call it a rough estimate.

- (e) We can just divide the flux by molar density of the solid to calculate the deposition rate: $\frac{dY}{dt} = \frac{J \cdot M_{\text{Si}}}{\rho_{\text{solid}}}$. At 10 cm, this will be 1.7 nm/sec; at 30 cm, 0.97 nm/sec. This means a 20 cm wafer placed 10 cm from the entrance (from $x = 10\text{cm}$ to $x = 30\text{cm}$) will have a 70% variation in deposition rate across it, which is a huge variation!
- (f) There are various ways to make the deposition more uniform. A common one is to place the substrates on an incline tilting upward about 30° , so the velocity increases with x , flattening out the concentration boundary layer a bit. One can also run under conditions more likely to be reaction-limited (*e.g.* at lower temperature), so the silane concentration in the gas will be roughly uniform throughout the chamber and right up to the wafers, making deposition roughly uniform. But the real answer is to ditch this reactor design, and go to something very different. Many modern reactors use stagnation flow produced by something like a shower head directing reactant gas at an individual wafer, or individual wafer rotation, to achieve near-uniform velocity and

concentration boundary layers. (In response to a similar question on last year's final, a student suggested "Rotisserie CVD" with wafers moving around like chickens in an oven...) Because it processes just a single wafer per chamber, this equipment is more expensive, but is worth it to deliver uniformity to within a few percent across each wafer.