PROFESSOR: All right, I feel almost as though I should introduce myself all over again. It's been a week and a half since we had a lecture. So let me begin by reminding you of what we were doing. We had derived all of the plane groups, and for better or worse had put them behind us. And then we moved into three dimensions, where things get a lot more involved and a lot more complicated, and to say the least, a lot more numerous. And the first question we asked was to say, when we're in a three-dimensional space, we can combine a first rotation operation with a second rotation operation, B beta If we're to begin by deriving point groups-- that is to say, the least [INAUDIBLE] point in this three-dimensional space is not going to move. [INAUDIBLE] two axes to intersect a point. For a space group, they could be parallel to one another. But that's gonna be an infinite set of symmetry elements and operations that extends through all space.

Then we asked the question-- rhetorically, because you knew I was going to answer-- what is the net result of a sequence of rotating from a first object to a second, and then picking up the second and rotating it to an angle beta about second axis. Begin the third one here. What is the net effect? And again, we could do that by the process of elimination. It has to be either translation or another rotation, because these are the only two generic sorts of operations which leaves the chirality unchanged. And I think I convinced you that indeed there was some third axis, C, which rotated directly from the first to the third by some different angle, gamma.

So that is the consequence of combining the first two rotation axes. What they would anticipate is that the location and also the value of the rotation [INAUDIBLE] depends on alpha and beta, and also [INAUDIBLE] at which we combine them. And that's what we'll see. You could do this in a number of ways that if you don't [INAUDIBLE] that gamma would turn out to be a crystallographic rotation. And then your result would be true, but it would not be a rotation operation which could exist in a three-dimensional point group.

So using the genius of Leonhard Euler and a construction known as Euler's construction. We set up a little spherical triangle, which we could analyze. Let me tell you a little bit about Euler, because he's a remarkable individual. The first remarkable feature of Euler is that he's Swiss, and there are not many world-class famous people who are Swiss, simply because the population is so small. The probability of somebody rising to heights is constant among all populations. If you have a small population, there are not going to be very many. And to demonstrate that point, can somebody identify some other citizen of Switzerland who rose to great heights, as world-famous as [INAUDIBLE]? Think of one other person?

I'm fairly pressed to do so myself. There's my uncle, but he actually didn't amount to much. But there's an artist, Paul Klee, who is world-class. He was one of the early modern artists. And Switzerland has just finished constructing a marvelous museum on the outskirts of the capital city, Berne. It's a structure that is supposed to mimic the rolling countryside of the central part of Switzerland. So it's a series of cylindrical structures, glass in front, glass on top. And it divides the area into three. Two of them are exhibit spaces and one is a space for scholars and researchers. And it's an absolutely marvelous structure. It appeared in the pages of *Time Magazine* when it opened [INAUDIBLE] about two months ago.

Anyway, Euler was born in 1707, so he operated a long time ago. And he died in St. Petersburg on September 18, 1783. That is exactly 350 years and one day after my birthday. That's another remarkable thing about him. He was 76 years old when he died. And that time of primitive medicine and plague, not many people got to live to their 60s and 70s. Euler studied at the University of Basel under the Bernoullis. I think you've all heard of the Bernoullis. There's a very famous principle of physics known as the Bernoulli effect, which stated in its simple practical form says that if you have the Sunday paper on the front seat alongside of you, and you drive your car with the windows down, the paper will blow out the window. That's Bernoulli's principle in action.

Euler got his doctorate from Basel at age 16. It sort of leads one to the rhetorical question, how come you guys have been spinning your wheels for so long? But then

I said, he was an unusual individual. The Bernoullis went to St. Petersburg in Russia, under Catherine the First. Russia was trying very hard at that time to enter the ranks of the Western world as a full member. Euler followed them a little bit later on. And he succeeded one of the Bernoullis as a professor of mathematics in 1733.

Then unfortunately, two years later in 1735, he lost the sight of one eye. And why? Because at that time, astronomy had been using this newfangled telescope which had recently been perfected. One of the hottest things going was studying the heavens looking through a telescope. And if you wanted to look at the sun, people knew nothing about the damaging effect on retinas of the sun's rays. So he lost the sight of one eye.

1741, he went back to Europe again, to Berlin. Why? Because the reigning monarch in Russia at that time was called Ivan the Terrible, and that says reams about why Euler would want to get out of Russia. But then 1776, he went back to St. Petersburg under the next monarch, who was Catherine the Great. And that says why one would be interested to go back. Finally, in 1766, he went fully blind. Did that slow him down? Not one bit.

He published in his lifetime 800 papers. You talk to some big cheese around MIT, they've published maybe 200 or 300 papers, and that with the assistance of an army of graduate students, and also, one might add, the assistance of Xerox machines and word processors. So back in the days when you wrote everything out by hand with a [INAUDIBLE] quill pen, 800 papers is an absolutely unbelievable accomplishment. It took 35 years after he passed away to publish everything that he'd written. People had kept busy publishing what he did.

And among his accomplishments, he was one of the first people to apply real hardcore mathematics to astronomy, to make it quantitative. He was one of the first to suggest that light was a wave form, and that color was a function of wave length. That was astonishingly precocious. And then, lest he seem like an egghead who spent all his time staring through telescopes and working out theorems to use in crystallography, he also wrote a popular account of science for the general public, which was published in 1768. And that book was published for 90 years, three generations of people kept gobbling up [INAUDIBLE] pretty good.

He impinged upon our own language and activities in several important ways. He was the one who used lowercase i to define the square root of minus 1. We can thank Euler for that. He was the person who used e to define the constant, 2.71828182845904523536. And he was the person who first used f to stand for function. So he contributed not only a lot of good mathematics, but a lot [INAUDIBLE].

So this does not have to be easy. Euler was a great guy. And this geometry of rotations about different axes is something that also survives in a mechanism that involves achieving angular core rotation on a axis by rotation on two orthogonal arcs. And that's something that's called an Euler Cradle.

And that is geometry that;s used in a great number of mechanical devices. In any case, back to instruction for our purposes. The thing that we would like to do is let alpha and beta take on all possible crystallographic values, namely 360 degrees or onefold axis, although we know that that's not gonna work. Twofold, 180 degrees. Threefold, 120. Fourfold, 90. Sixfold, 60. And let that give the values to alpha and beta, taking two at a time.

And then let us ask the question, at what angle should we combine these two axes to get gamma to be a crystallographic rotation axis? And if it is crystallographic and not something like 37.9234 degrees, what are the remaining axes with interaxial angles B and A? So this is the problem that Euler's construction solved. And I won't go through all the arguments that we need to set this up. But what we found after some f sleight of hand when we were working on the polar triangle with spherical trigonometry, what we found was the result that said that if we want to combine two axes, alpha and beta, so that the third one turned out to be a rotation of gamma, then the cosine of the angle between A and B should be the cosine of alpha/2, cosine of beta/2 plus cosine of gamma/2, divided by the sine of alpha/2, sine beta/2.

So if you pick your alpha and beta, and you decide what you would want these first two rotations to turn out to be. And generally it's not gonna work. But there are a surprising number of cases where it does work. So you specify the combination you had. You also determine the angle between A and C. And you have to also determine the angle between the axes B and C. And there are similar sorts of expressions that one obtains simply by [INAUDIBLE] alpha and beta again. So then we set it up just by looking at all possible combinations of twofold, of two different rotation axes, and a third, which the net effect might be. We're not interested in permuting A, B, and C. And A equal to C equal to B is just as interesting or not as A equal to B equal to C. We don't care about permutations.

And we generated-- just as we [INAUDIBLE] a week and a half ago-- a set of combinations that we should consider, what the axis A would be, what the axis B would be, and them different choices for the axis C. So A could be 1. B could be 1. And we could look for 1, 1, 1; 1, 1, 2; 1, 1, 3; 1, 1, 4; 1, 1, 6. Those are legitimate combinations? Those are absurd combinations, because doing nothing about the first onefold axis, doing nothing about the second onefold axis could hardly result in the net effect of the 90-degree rotation by the third axis. And [INAUDIBLE] suggested is that sitting around and doing nothing twice was equal to a rotation [INAUDIBLE] its junctures [INAUDIBLE] we'd find ourselves spinning on our axis like tops.

Twofold axis. 1, 1, 2, we have here, so we don't have to consider 2, 1, 1. But we should consider 2, 1, 2; 2, 1, 3; 2, 1, 4; 2, 1, 6, and so on. If we filled out this whole table, last time you got a copy of it and some notes, and all that remains then is to quote [INAUDIBLE] and see where we get allowable axial combinations. And not surprisingly, there are so very, very few. And we showed-- again, when we finished up last time-- that you can always take any n-fold axis that has a C gamma that's equal to C 2 pi over n and combine it with twofold axes at right angles, provided you make the angle between the twofold axes equal 2 of gamma over 2.

So the crystallographic possibilities for C are, first of all, C could be a twofold axis, in which case you could combine with it a pair of twofold axes, and this 1/2 of 180 degrees would also be a right angle. We could let C be a threefold axis, in which case the twofold axes have to be at right angles, two- to threefold axes. And they should be combined at half the angular throw of the threefold axes, which is 60 degrees. Two more possibilities are four [INAUDIBLE] pair of twofold axes at right angles. Of the angle between them, half of the throw of a fourfold axis would have to be equal to 5 degrees. And the last one is sixfold axis with a pair of twofold axes. Add angles to it. And a third [INAUDIBLE].

So notice the insidious fact that the angle between the twofold axes is always a half, 1/2 the rotation angle in principal axis C are not equal to this rotation axis. The other thing we saw that is that these twofold axes are different, distinct, symmetry-independent axes. They're different in that the principal axis C never rotates this axis into the second one, and therefore demands that whatever's going on around one twofold axis be identical to what's going on at the other twofold axis, different in the sense that they function in different ways in the pattern, or if you're describing the symmetry of an object. So probably the best demonstration of this is a regular prism with a triangular shape or with a square shape or with a hexagonal shape. And the adjacent twofold axes for these prisms would come out of the normal to a face.

And then if the second twofold axis is going to be 45 degrees away from the first, the other one has to come [INAUDIBLE]. So yeah, they function in different ways in the space. One is normal to the face of a regular prism, if that's what's in our space. The other one comes out of the edge. Similarly for a sixfold axis, a hexagonal prism has one twofold axis coming out normal to a face, the adjacent twofold axis coming out to an edge. And as advertised, that angle is 33. So they function in different ways.

The only exception to that, again, is the [INAUDIBLE] threefold axis. And the twofold axes there, which were 60 degrees, come out of one side from a corner of the triangular prism. On the other side, that was [INAUDIBLE]. So all of the twofold axes were the same thing. There's only one independent kind of twofold axis, just as there was only kind of mirror plane in the combination of a mirror plane passing through a twofold axis. The names for these are always, as we've done in the past, a running list of the independent symmetry operations that are present. So this general one, n, 2, 2, with the n-fold axis for some generic sort of a twofold axis. This one would be 2, 2, 2. This one would be 4, 2, 2. And this one would be 6, 2, 2.

This [INAUDIBLE] with a threefold axis is, again, called not 3, 2, but 3, 2, because there's only one kind of twofold axis, just as there's only one kind [INAUDIBLE]. We pause [INAUDIBLE], see if you have any questions. These are the crystallographic combinations of this [INAUDIBLE]. There is no reason why you should not in something that doesn't have to be compatible with a lattice, combine an n-fold axis of any sort with twofold axes or right angles.

And indeed, if I look at my old friend, the saguaro cactus, we can add anything like 28- up to 32-fold symmetry. This cactus stem had a 28-fold symmetry. It would be a twofold axis coming out of the string with one of the ribs, another twofold axis coming out of the crevice between the pair of these ribs. And if I took that thing up, very carefully because of the spines, and with great effort because it weights several tons, and rotated it about one axis, and then rotate it again about a second axis, turning it upside-down, [INAUDIBLE] axis would be rotation to 128 [INAUDIBLE]. Valid symmetry, but not crystallographic.

OK, any comments or questions? Get to know these results, because the exercise that's going to occupy us for the next week is going to be asking how we can decorate these frameworks with mirror planes and with inversion. If you want orientation, we could add another operation to the collection of axes while it pops up. Where [INAUDIBLE]. Comments or questions? OK, there are only two other combinations that are crystallographic that involve directions that are not simple. And one of them involves a combination of a threefold axis with twofold axes that come out of directions at a normal to the face of a cube. So these turn out to be very, very strange angles which make no sense at all until you refer them to directions in the cube.

The direction of the threefold axis turns out to be correspondent with the angle of a cube. The direction of the twofold axis corresponds to the normal two faces. You can show-- I did show, and I don't think anybody really followed me, so I had to hand out that [INAUDIBLE]-- that if you start with one twofold axis and one threefold axis, what you're going to get is a threefold axis coming out of all of the [INAUDIBLE] diagonals, but they're all equivalent to this threefold axis.

And twofold axes come out to normal for all the faces of the cube. So there's only one kind of twofold axis and one kind of threefold axis, so even though we got this by combining a pair of twofold axes-- I'm sorry, a pair of threefold axes-- that's what happens when you stay up late. You forget about this one. A pair of threefold axes at the diagonals and one twofold axis. And this is the combination that is called 23, because there's only one sort of twofold axis and one sort of threefold axis.

And I'd like to point out and I'd like to warn you of traps when we come across them, make sure we don't [INAUDIBLE] across them. Notice the insidious relation of this pair of integers to the symmetry that we label [? 232. ?] 32 is a threefold axis with a twofold axis normal to it. 23 is this combination that involves corrections in a cube.

Now, let me pause parenthetically with an aside. You might say, how can this be? Here is a cube. That cube has got a fourfold axis about it. Don't call that a twofold axis. A cube has a fourfold axis coming out of it. OK, let me give you an example in real life. There is a fairly common mineral, iron disulfide-- pyrite. This forms nice, shiny cubes, but the cube faces have striations on them.

If you look at them, there's a set of lines running this way. And what those lines are if you look at this crystal face with a magnifying glass, is that these are little steps of a second face. And this is a face of the form hk0. And this oscillates back and forth, and there seem to be lines scribed on the surface. So this sort of a [INAUDIBLE] crystal growth of one face which never really develops is not that uncommon. There's a threefold axis coming out of the corner here, but this is really a surface that is left at variant only by 183 [INAUDIBLE]. You cannot rotate that surface 90 degrees. The orientation of the lines have changed.

But there is a bona fide threefold axis coming up there, so these striations, if I rotate them to this face, will go in a way like this. This edge turns into this edge, and therefore the lines will run down like this. And if I rotate [INAUDIBLE] n by 90 degrees, the striations on the adjacent face will run down. So there's a decorated cube. And if you rolled it up and say how can I move this cube around and leave its appearance totally unchanged? the answer is, rotate it by 120 degrees [INAUDIBLE] diagonal. But we can only rotate it 180 degrees around the face. So there's an example of a crystal [INAUDIBLE] on the arrangement of rotation axes 23.

The final one, the highest symmetry of all, involves a fourfold axis coming out of a direction normal to a face, a threefold axis coming out of the [INAUDIBLE] diagonal, and a twofold axis coming out normal to an edge. And if you let these axes work on one another, there's a twofold axis that comes out of every edge, and a fourfold axis that comes out of every face. And this one is named 432, because it's a combination of a fourfold, a threefold, and a twofold. That is the symmetry to the cube. And for crystallographic symmetries, that's about as complicated as it gets.

Now, if we look at the regular solids that we've encountered here, with symmetry 23, there is a regular [INAUDIBLE] consisting of four triangular faces. That's a tetrahedron. And for 432, one of the polyhedra that can form from the crystal [INAUDIBLE] is an octahedron. These were the lovely solids called Platonic solids, after Plato, that we used as our trophies early this afternoon. So this is an octahedron. Let me finish up before our break by asking is there any other regular polyhedra that can result from a combination of rotation axes that are not crystallographic? Now, that's a tough question to ask. You instantly scan your knowledge of geometry.

Clearly, there are a lot of prisms [INAUDIBLE] infinite number of prisms [INAUDIBLE] n22. But there's only one other combination of axes non-crystallographic which results in a regular polyhedron. And this is a combination, believe it or not, of a fivefold axis with a threefold axis. This is a fivefold axis and a threefold axis and a twofold axis. And these result in a regular solid called an icosahedron. And that is so complicated that I won't attempt to draw it. But having said so, it looks like this. It has diamond-shaped faces. And there are five of these that come together in a [INAUDIBLE] form.

So here's one of the [INAUDIBLE] diamond-shaped faces, another diamond-shaped face, and then the two other ones that come in like this. So there are 1, 2, 3, 4, 5 faces, so this is the orientation of a fivefold axis. The twofold axis comes out of a place where two of these edges are shared. And the threefold axis-- [INAUDIBLE] triangular faces. [INAUDIBLE].

So here's the fivefold axis. These are the twofold axes [INAUDIBLE]. And these are threefold axes. But you know all this. Is there anybody who [INAUDIBLE] show me that they've never seen an icosahedron? Anybody ever who has not seen-- have you seen it? [INAUDIBLE]. Here-- I spared no expense-- is a live icosahedron for those who would like to look.

AUDIENCE: How many sides does it have?

PROFESSOR: He'll count them for you. I don't remember. I know the number of faces and the number of edges, but not the [INAUDIBLE].

AUDIENCE: [INAUDIBLE] faces.

PROFESSOR: I think it's-- I'm not sure [INAUDIBLE]. Not sure.

AUDIENCE: But you said you know the number of faces.

PROFESSOR: Yeah, but I can't tell you everything I know [INAUDIBLE].

AUDIENCE: Yeah, yeah.

PROFESSOR: There's another figure which also has this symmetry, and it's a regular solid. And this is called a rhombic dodecahedron. And, wise guy, this has 12 faces. And the faces are pentagonal. And there are three pentagonal faces that come together at the threefold axis, a fivefold axis out of each of the pentagonal faces, and a twofold axis out of the edges.

The face that this has 12 faces at regular intervals leads an entrepreneur who was familiar with injection molding to cast these little things as a plastic, and then puts a month of the year on each of the 12 faces and made a nice little desk calendar.

[INAUDIBLE] something that reminds you of symmetry as well [INAUDIBLE]. So this has fivefold faces, 12 of them, and that's the rhombic dodecahedron as opposed to the normal dodecahedron which is [INAUDIBLE].

AUDIENCE: So are those both [INAUDIBLE]?

PROFESSOR: I'm sorry?

AUDIENCE: Are those both [INAUDIBLE]? The icosahedron, [INAUDIBLE]?

PROFESSOR: Yeah, this has a fivefold axis coming out of the pentagonal [INAUDIBLE]. Is that what you're asking? And the threefold axis, there are three of them that, come together, and they do something like this. So here's the threefold, here's the fivefold, here's the twofold. Here we have to see it [INAUDIBLE]. Look for somebody who's got one of these desk calendars. [INAUDIBLE] used to sell them. [INAUDIBLE].

OK, this sets up the next stage of our game. We've got these arrangements of axes. And if you count them up on the fingers of your hands and one toe, there are 11 of them. There are the axes by themselves-- onefold, twofold, threefold, fourfold, sixfold. There are these so-called dihedral combinations, where the only thing that changes from one to the other is the symmetry of the main axis, the [INAUDIBLE] symmetry, and therefore the dihedral angle between the twofold axes.

And these are 222, 32, 422, and 622. And then the two cubic arrangements, 32 and 432. So when we return, we'll ask the question, how can we add mirror planes for an inversion center to this combination of axes? And these are going to give us new symmetries involving not only rotation, but [INAUDIBLE] help with rotation inversion as well.

And the constraint in doing this is that we have to add the reflection plane and the inversion center in such a way that it doesn't create any new rotation axes, because we have systematically derived all of the possible combinations of crystallographic rotation axes. So if the addition of a mirror plane, creates a new axis, that's going to be something that you already have with a combination of a greater number of rotation axes. For example, if you take a single twofold axis and a mirror plane of an angle, that generated another twofold axis 90 degrees away. But we've already got that.

If a mirror plane moves an axis to an angle that doesn't correspond to one of these arrangements, it's going to be impossible, because we have systematically derived, using Euler's construction, all the combinations of rotation operations that are possible. So that's going to be the constraint. We want to add mirror planes or an inversion center in all possible combinations. And this means we're gonna need a theorem.

What happens when you add a mirror plane to a rotation operation? We're already familiar with one of them. You take an axis and you pass an mirror plane through it, you get another mirror plane that is rotated about the axis, away from the first, by half the rotation angle of the axis. So let's take a breather, and let us resume in about 10 minutes.