Solutions to Problem Set 7

Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding. You may not copy another student's solutions. And you should not refer to notes from a study group while writing up your solutions (if you need to refer to notes from a study group, it isn't really "your own understanding").

Part I. These problems are mostly from the textbook and reinforce the basic techniques. Occasionally the solution to a problem will be in the back of the textbook. In that case, you should work the problem first and only use the solution to check your answer.

Part II. These problems are not taken from the textbook. They are more difficult and are worth more points. When you are asked to "show" some fact, you are not expected to write a "rigorous solution" in the mathematician's sense, nor a "textbook solution". However, you should write a clear argument, using English words and complete sentences, that would convince a typical Calculus student. (Run your argument by a classmate; this is a good way to see if your argument is reasonable.) Also, for the grader's sake, try to keep your answers as short as possible (but don't leave out *important* steps).

Part I(20 points)
    (a) (2 points) p. 318, Section 9.5,   Problem 26
    (b) (2 points) p. 348, Section 10.4, Problem 10
    (c) (2 points) p. 348, Section 10.4, Problem 13
    (d) (2 points) p. 348, Section 10.4, Problem 14
    (e) (2 points) p. 350, Section 10.5, Problem 10
    (f) (2 points) p. 356, Section 10.6, Problem 11
    (g) (2 points) p. 356, Section 10.6, Problem 12
    (h) (2 points) p. 362, Section 10.7, Problem 14
    (i) (2 points) p. 362, Section 10.7, Problem 22(a)
    (j) (2 points) p. 362, Section 10.7, Problem 22(b)

Solution to (a) Since $\tan \alpha = \frac{a}{x}$ and $\tan \beta = \frac{a+b}{x}$, we obtain

$$\theta = \beta - \alpha = \boxed{\tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{a}{x}\right).}$$

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To locate when θ takes on its extrema, we differentiate θ with respect to x and set the result to zero: $\frac{d\theta}{dx} = \frac{d}{dx}\tan^{-1}\frac{a+b}{x} - \frac{d}{dx}\tan\frac{a}{x} = \frac{1}{(\frac{a+b}{x})^2+1}\frac{-(a+b)}{x^2} - \frac{1}{\frac{a^2}{x^2}+1}\frac{-a}{x} = \frac{a}{a^2+x^2} - \frac{a+b}{(a+b)^2+x^2}$. The angle θ takes on its maximum value when $\frac{d\theta}{dx} = 0$, i.e. when $\frac{a}{(a^2+x^2)} = \frac{a+b}{(a+b)^2+x^2}$, or equivalently, $x^2 = a(a+b)$, that is $\boxed{x = \sqrt{a(a+b)}}$.

Solution to (b) The radical $\sqrt{x^2 - a^2}$ suggests the change of variables $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$ which yields

$$
\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a^3\sec^3\theta a\tan\theta}d\theta = \frac{1}{a^2}\int \cos^2\theta d\theta = \frac{1}{a^2}\int \frac{1}{2}(1+\cos 2\theta)d\theta
$$
$$
= \frac{1}{a^2}(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta) + C = \frac{1}{a^2}(\frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta) + C \tag{1}
$$
$$
= \boxed{\frac{1}{2a^3}\cos^{-1}(\frac{a}{x}) + \frac{\sqrt{x^2-a^2}}{2a^2x^2} + C}
$$

Solution to (c) The term $a^2 - x^2$ suggests the substitution $x = a\sin\theta$, and this leads to integration of $\sec\theta$ at the end. While this is okay, as long as you know that $\int \sec\theta d\theta = \ln(|\sec\theta + \tan\theta|) + C$, there are easier ways: we can use either partial fractions, or inverse substitution $x = a\tanh t$. Method of partial fractions proceeds as follows:

$$
\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}
$$

The Heaviside method (or multiplying out and letting $x = a$, $x = -a$) yields, respectively, $\frac{1}{2a} = A$ and $\frac{1}{2a} = B$. Straightforward integrating, we obtain

$$
\int \frac{1}{4}(\frac{1}{a-x} + \frac{1}{a+x}) = \boxed{\frac{1}{2a}\ln|\frac{a+x}{a-x}| + C}
$$

Solution to (d) The radical $(a^2 - x^2)^{-3/2}$ suggests the substitution $x = a\sin\theta$, $dx = a\cos\theta d\theta$, which leads to

$$
\int (a^2-x^2)^{-3/2}dx = \int (a^2\cos^2\theta)^{-3/2}a\cos\theta d\theta = \frac{1}{a^2}\int \sec^2\theta = d\theta
$$
$$
= \frac{1}{a^2}\tan\theta + C = \boxed{\frac{x}{a^2\sqrt{a^2-x^3}}} \tag{2}
$$

Solution to (e) We first need to work on the radical $\sqrt{x^2 + 2x - 3}$. In an attempt to complete the expression under the square root to a perfect square, we notice that $x^2 + 2x - 3 = x^2 + 2x + 1 - 4 = (x+1)^2 - 4$. This inspires the substitution $u = x + 1$, $du = dx$. Then our integral becomes

$$
\int \frac{\sqrt{x^2+2x-3}}{x+1}dx = \int \frac{\sqrt{u^2-4}}{u}du
$$

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which, because of the radical $\sqrt{u^2 - 2^2}$ is screaming for the change of variables $u = 2\sec\theta$, $du = 2\sec\theta\tan\theta d\theta$. Then our integral equals

$$\int \frac{2\tan\theta}{2\sec\theta} 2\sec\theta\tan\theta d\theta = 2\int \tan^2\theta d\theta = 2\int (\sec^2\theta - 1)d\theta$$

$$= 2\tan\theta - 2\theta + C = \sqrt{u^2 - 4} - 2\cos^{-1}(\frac{2}{u}) \tag{3}$$

$$= \boxed{\sqrt{x^2 + 2x - 3} - 2\cos^{-1}(\frac{2}{x+1}) + C}$$

Solution to (f) As the nominator is a second order polynomial and the denominator is a third order one, we can directly proceed into decomposing this rational expression into its partial fractions: First we factor the denominator $x^3 + 2x + x$. The factor x is obvious: $x^3 + 2x + x = x(x^2 + 2x + 1)$ and the remaining part is just $(x + 1)^2$. Therefore we can write

$$\frac{-4x^2 - 5x - 3}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

The Heaviside cover-up method goes as follows: Covering terms without $1/x$ and assigning $x = 0$, we obtain $\frac{-4x^2 - 5x - 3}{(x+1)^2} = \frac{0+0-3}{1^1} = A$, i.e $A = -3$. Similarly, covering terms without $1/(x + 1)^2$ and assigning $x = -1$ yields $\frac{-4+5-3}{(-1)} = 2 = C$. The other constant B can not be determined from Heaviside method- remember, Heaviside method does not give all the coefficients when there is a repeated factor, like here we have $(x + 1)^2$. So how do we find B? I am glad you asked: the easiest way is to plug in some arbitrary value for x. To keep the algebra simple, we put $x = 1$ in

$$\frac{-4x^2 - 5x - 3}{2^2} = \frac{-3}{x} + \frac{B}{x + 1} + \frac{2}{(x + 1)^2}$$

This yields $B = -1$. Therefore

$$\int \frac{-4x^2 - 5x - 3}{x(x + 1)^2} dx = \int -\frac{3}{x} + \frac{-1}{x + 1} + \frac{2}{(x + 1)^2} dx = \boxed{-3\ln|x| - \ln|x + 1| - \frac{2}{x+1} + C}$$

Solution to (g) We decompose the given rational integrand into partial fractions as

$$\frac{4x^2 + 2x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{B}{x^2 + 4} + \frac{Cx}{x^2 + 4}$$

Multiplying out yields $4x^2 + 2x + 4 = Ax^2 + 4A + Bx + Cx^2$. Equating the coefficients of x^2,x and 1 respectively, we obtain $A = 4 + C$, $2 = B$, and $4 = 4A$, hence $A = 1$. Therefore $C = 3$ from the first equation. Using the fact that

$$\frac{d}{dx}\tan^{-1}(\frac{x}{a}) = \frac{a}{a^2 + x^2}$$

to integrate the second term and making the easy substitution $u = x^2 + 4$ to integrate the last term, we obtain

$$\int \frac{4x^2 + 2x + 4}{x(x^2 + 4)}dx = \int \frac{1}{x} + \frac{2}{x^2 + 4} + \frac{3x}{x^2 + 4}dx = \boxed{\ln x + \tan^{-1}(\tfrac{x}{2}) + \tfrac{3}{2}\ln|x^2 + 4| + D}$$

Solution to (h) Let $u = \sin(\ln x)$, $dv = dx$, then $du = \frac{1}{x}\cos(\ln x)$ and $v = x$. The integration by parts formula then yields

$$\int \sin(\ln x)dx = \sin(\ln x)x - \int x\frac{1}{x}\cos(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

We need only one more integration by parts, this time with $u = \cos(\ln x)$, $du = -\frac{1}{x}\sin(\ln x)$, $dv = dx$, $v = x$, to obtain

$$\int \sin(\ln x)dx = x\sin(\ln x) - [x\cos(\ln x) - \int x\frac{1}{x}(-\sin(\ln x))dx]$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx \tag{4}$$

Rearranging this and dividing by two, we obtain

$$\int \sin(\ln x)dx = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$$

Solution to (i) We write

$$\int \cos^n(x)dx = \int \cos^{n-1}(x)\cos(x)dx$$

and implement integration by parts with $u = \cos^{n-1}(x)$, $dv = \cos(x)dx$, and $du = (n-1)\cos^{n-2}(x)\sin(x)dx$, $v = \sin(x)$. Therefore,

$$\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + \int (n-1)\cos^{n-2}(x)\sin^2(x)dx$$

$$= \cos^{n-1}(x)\sin(x) + \int (n-1)\cos^{n-2}(x)(1 - \cos^2(x))dx \tag{5}$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx$$

Rearranging this equation, we obtain the sought *recursion formula*

$$\int \cos^n(x)dx =$$
$$\boxed{\tfrac{1}{n}\cos^{n-1}(x)\sin(x) + \tfrac{n-1}{n}\int \cos^{n-2}(x)dx}$$

Solution to (j) Using our result to **(i)**, we can write

$$\int_0^{\pi/2} \cos^n(x)dx = (\frac{1}{n}\cos^{n-1}(x)\sin(x)|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-1}(x)dx)$$

Observe that $\cos^{n-1}(x)\sin(x)|_0^{\pi/2}$ is zero for $n \geq 2$. Let us call

$$A_n = \int_0^{\pi/2}\cos^n(x)dx.$$

Combined with this observation, the recursion formula of **(i)** gives

$$\begin{aligned} A_n &= \frac{n-1}{n}A_{n-2}, \quad n \geq 2 \\ &= 1 \qquad\qquad n = 1 \\ &= \pi/2 \qquad\quad n = 0 \end{aligned} \tag{6}$$

Here the values have been found directly for $n = 0$ and $n = 1$. Therefore,

$$A_7 = \frac{6}{7}A_5 = \frac{6}{7}\frac{4}{5}A_3 = \frac{6}{7}\frac{4}{5}\frac{2}{3}A_1 = \boxed{16/35.}$$

Part II(30 points)

Problem 1(15 points) This problem sketches a systematic method for finding antiderivatives of expressions $F(\sin(\theta), \cos(\theta))$, where F is a fraction of polynomials.

(a)(8 points) With the substitution $\tan(\theta/2) = z$, verify that,

$$\sin(\theta) = \frac{2z}{1+z^2}, \quad \cos(\theta) = \frac{1-z^2}{1+z^2}, \quad d\theta = \frac{2dz}{1+z^2} \tag{7}$$

Please try this on your own first. However, if you are very stuck, take a look at Problem 5E-12 in the course reader.

Solution to (a) Using the double angle formula for tangents, we have $\tan\theta = \frac{2\tan(\frac{\theta}{2})}{1-\tan^2(\frac{\theta}{2})}$ We then draw the right triangle with two right sides having length $2z$ and $1 - z^2$. Pythagorean theorem yields the hypotenuse: $h^2 = (1-z^2)^2 + (2z)^2 = 1 + 2z^2 + z^4 = (1+z^2)^2$.Therefore the hypotenuse has length $h = 1 + z^2$. Now, we can easily read off the values

$$\sin\theta = \frac{2z}{1+z^2}, \quad \cos\theta = \frac{1-z^2}{1+z^2}$$

In addition, $dz = d(tan(\frac{\theta}{2})) = \sec^2(\frac{\theta}{2})\frac{d\theta}{2}$, therefore

$$d\theta = 2\cos^2(\frac{\theta}{2})dz = (1+\cos\theta)dz = (\frac{1-z^2}{1+z^2} + \frac{1-z^2}{1+z^2})dz = \frac{2dz}{1+z^2} \tag{8}$$

(b)(7 points) By Example 3 on p. 571 of the textbook, the polar equation of the conic section with *eccentricity e* and *focal parameter p · e* is,

$$r = f(\theta) = \frac{ep}{1 - e\cos(\theta)}$$

Set up the integral for the area of the region bounded by $0 \leq r \leq f(\theta)$ for $a \leq \theta \leq A$. Use **(a)** to turn this into an integral involving only $z = \tan(\theta/2)$. Use the nonnegative constant,

$$b = \sqrt{\frac{|1 - e|}{1 + e}},$$

to simplify your answer. You need not evaluate the integral.

Solution to (b) Remembering the area formula for curves in polar coordinates, the required area is

$$Area = \int_a^A \frac{r^2}{2}d\theta = \int_a^A \frac{e^2p^2}{(1 - e\cos\theta)^2}d\theta$$

We now plug in the substitution given by (7) and obtain

$$Area = \int_{\tan(a/2)}^{\tan(A/2)} \frac{e^2p^2}{(1 - e\frac{1-z^2}{1+z^2})}\frac{dz}{1+z^2}$$
$$= \frac{e^2p^2}{(1+e)^2}\int \frac{1+z^2}{(\frac{1-e}{1+e} + z^2)^2}dz \qquad (9)$$

Using the suggested notation $\pm b^2 = \frac{1-e}{1+e}$, we separate the two cases depending on the sign of the expression $\frac{1-e}{1+e}$. If $e < 1$, then expressing the nominator $z^2 + 1 = z^2 + b^2 - (b^2 - 1)$ we have

$$Area = \boxed{\frac{e^2p^2}{(1+e)^2}\left[\int_{\tan(a/2)}^{\tan(A/2)} \frac{1}{b^2+z^2}dz - (b^2 - 1)\int_{\tan(a/2)}^{\tan(A/2)} \frac{1}{(b^2+z^2)^2}dz\right] \text{ if } 0 \leq e \leq 1.}$$

In case $e > 1$, we express the numerator $z^2 + 1 = z^2 - b^2 + (b^2 + 1)$, and hence

$$Area = \boxed{\frac{e^2p^2}{(1+e)^2}\left[\int_{\tan(a/2)}^{\tan(A/2)} \frac{1}{z^2-b^2}dz - (b^2 + 1)\int_{\tan(a/2)}^{\tan(A/2)} \frac{1}{(z^2-b^2)^2}dz\right] \text{ if } e > 1.}$$

Problem 2(15 points)

(a)(5 points) Read through Section 10.3 (it is only 3 and a half pages). Everybody will get credit for this part.

Solution to (a) We first get our hands on the textbook by Simmons. To locate section 10.3, we use the "contents" part of the book. We see that the section starts on page 340. We now open page 340 and read until the end of the section.

(b)(5 points) Use **Problem 1 (a)** to rewrite the integral,

$$\int \sin^m(\theta)\cos^n(\theta)d\theta,$$

as an integral involving only $z = \tan(\theta/2)$.

Solution to (b) Again, plugging the substitutions in (7) yields

$$\int \sin^m\theta \cos^n\theta d\theta = \int (\frac{2z}{1+z^2})^m(\frac{1-z^2}{1+z^2})^n\frac{2dz}{1+z^2} =$$

$$\boxed{\int 2^{m+1}z^m(1-z^2)^n/(1+z^2)^{m+n+1}dz.}$$

(c)(5 points) Assume $n \geq 2$. Use the following integration by parts,

$$\int \sin^m(\theta)\cos^n(\theta)d\theta, \ u = \sin^m(\theta)\cos^{n-1}(\theta), \ dv = \cos(\theta)d\theta,$$

to find a reduction formula of the form,

$$x\int \sin^m(\theta)\cos^n(\theta)d\theta = F(\sin(\theta), \cos(\theta) + A \int \sin^{m+2}(\theta)\cos^{n-2}(\theta)d\theta.$$

Solution to (c) One requirement in integration by parts is to choose u and dv so that we can integrate dv easily to get v. In this problem, one such choice would be $dv = \sin^m\theta\cos\theta d\theta$, so that $v = \frac{1}{m+1}sin^{m+1}\theta$ and the remaining part is $u = \cos^{n-1}\theta$. Then $du = -(n-1)cos^{n-2}\theta\sin\theta$. All these with the integration by parts formula yield

$$\int \sin^m\theta\cos^n\theta d\theta = \boxed{\frac{1}{m+1}sin^{m+1}\theta\cos^{n-1}\theta + \frac{n-1}{m+1}\int \sin^{m+2}\theta\cos^{n-2}\theta d\theta}$$

Alternatively, you can try $dv = \cos\theta d\theta$ and $u = sin^m\theta\cos^{n-1}\theta$. It works, but is a little bit longer.

Not to be turned in: Of these three methods for evaluating the integral, which would be easiest to use and fastest on an exam?