18.01 Single Variable Calculus
Fall 2006

18.01 Problem Set 6

Due THURSDAY 11/09/06, 12:55 pm

Warning: This problem set is due on a **THURSDAY** not Friday, because of the Veterans' Day holiday. It is due **before lecture**, which is at 1:05 on Thursday.

Even though this problem set is due two days after Exam 3, you will need to do most of it by Tuesday, in the process of preparing for Exam 3 — all except the Part I problems connected to Lecture 25.

Part I   (22 points)

Lecture 22. Fri. Oct. 27   Volumes by disks and shells.
     Read: 7.4   Work: 4B-2eg, 5; 4C-1a, 2, 3 4J-3

Lecture 23. Tues. Oct. 31   Work; average value; probability.
     Read: 7.7, to middle p. 247   Notes AV.
     Work: 249/5, 6, 15 (solutions posted at web site); 4D-2, 3, 5

Lecture 24. Thurs. Oct. 29   Numerical Integration.
    Read 10.9   Work: 3G-1ad, 4

Lecture 25. Fri. Nov. 3   Trigonometric integrals. Direct substitution.
    Read 10.2, 10.3   Work: 5B-9, 11, 13, 16; 5C-5, 7, 9, 11 (due after Exam 3)

Lecture 26. Tues. Nov. 7   **Exam 3 1:05-1:55** covering lectures 18–24.

Part II   (30 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 22, 7pts: 3 + 4) Do 7.4/12 and 13.

2. (Lec 23, 4pts) The voltage V of house current is given by

$$V(t) = C \sin(120\pi t)$$

where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

3. (Lec 23, 6 pts: $1 + 2 + 1 + 2$)

a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ with probability equal to the area.

b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ with probability **proportional** to the area. (Probability = Part/Whole).

c) Evaluate
$$W = \int_0^\infty e^{-at} dt = \lim_{N \to \infty} \int_0^N e^{-at} dt$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole."

The probability that a radioactive particle will decay some time in the interval $0 \leq t \leq T$ is

$$P([0, T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W} \int_0^T e^{-at} dt$$

Note that $P([0, \infty)) = 1 = 100\%$.

d) The half-life is the time T for which $P([0, T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$. Suppose that a radioactive particle has a half-life of 1 second. What is the probability that it survives to time $t = 1$, but decays some time during the interval $1 \leq t \leq 2$? (Give an integral formula, and use a calculator to get an approximate numerical answer.)

4. (Lec 24, 6pts) The basis for Simpson's rule is the following formula. Let x_1 be the midpoint of the interval $[x_0, x_2]$, and denote its length by $2h$. Consider any three points (x_0, y_0), (x_1, y_1), (x_2, y_2). There is a unique quadratic function (parabola)

$$y = Ax^2 + Bx + C$$

whose graph passes through the three points. Simpson's rule says that the area under the parabola above $[x_0, x_2]$ is

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

This problem is devoted to proving this formula. It is significant because it illustrates how calculations can be simplified by using symmetry, and by looking ahead to see what you need.

Since the area will be the same if the parabola is translated to the left or right, we may assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then in terms of the rest of the data (i.e., h and the y_i)

make a sketch and determine C;

show, by integrating, that to find the area we need only determine A (or better, $2Ah^2$);

determine $2Ah^2$ using the data;

put the results together to establish the formula for area.

5. (Lec 24, 4pts) Use a calculator to make a table of values of the integrand and find approximations to the Fresnel integral $\int_0^a \cos(t^2) dt$ for $a = \sqrt{\pi/2}$, using Simpson's rule with four and eight intervals. (The exact answer to five decimal places is 1.22505. Record your approximations to six decimal places to compare.)