<u>The small span theorem and the extreme-value theorem.</u>

There are three fundamental theorems concerning a function that is continuous on a closed interval $[a,b]$. The first is the <u>Intermediate-Value Theorem</u>, which is stated and proved on p. 144 of Apostol. We consider the other two here.

We begin with a definition.

<u>Definition.</u> If the function f is bounded on the interval $[c,d]$, we define the <u>span</u> of f on this interval as follows: Let

$$M(f) = \sup \{f(x);\ x \in [c,d]\},$$

$$m(f) = \inf \{f(x);\ x \in [c,d]\}.$$

Then we define the span of f by the equation

$$\text{span}\ (f) = M(f) - m(f).$$

<u>Theorem. (The small-span theorem). Let f be continuous on the closed interval $[a,b]$. Given $\varepsilon > 0$, there is a partition</u>

$$x_0 < x_1 < \dots < x_n$$

<u>of the interval $[a,b]$ such that f is bounded on each closed subinterval $[x_{i-1}, x_i]$, and such that the span of f on each closed subinterval is at most ε.</u>

<u>Proof.</u> The proof of this theorem is a bit tricky, but the theorem is so useful that the effort is justified.

One proceeds by what is sometimes called "the method of successive bisections," or less elegantly, "chopping the interval in half repeatedly"!

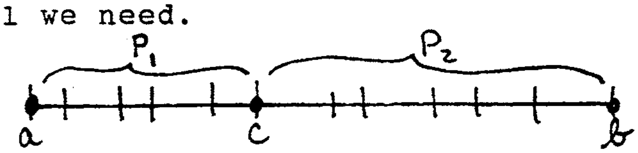
For purposes of this proof, let us make up some terminology. If f is defined on the interval [c,d], we shall say that

f <u>is</u> ε <u>-pleasant on the interval</u> [c,d]

if there is some partition of [c,d] such that the span of f on each closed subinterval of the partition is at most ε. If there is no such partition, we shall say that f is ε -<u>unpleasant</u> on [c,d]!

Our object then is to prove that if f is continuous on [a,b], then f is ε-pleasant on [a,b].

We make the following remark: Let c be any number with $a < c < b$. If f is ε-pleasant on [a,c], and if f is also ε-pleasant on [c,b], then f is ε-pleasant on all of [a,b]. The proof is easy. One merely takes the appropriate partitions of [a,c] and [c,b] and puts them together to get a partition of [a,b]. This simple fact is all we need.



We now prove the theorem. Assume that f is continuous on [a,b], and that f is ε-unpleasant on [a,b]. We shall derive a contradiction.

<u>First step.</u> Let c be the midpoint of [a,b]. Since f is ε-unpleasant on [a,b], it must be true that f is ε-unpleasant on either [a,c], or on [c,b], or on

both. Let $[a_1,b_1]$ denote the left half $[a,c]$ of our interval if f is ε-unpleasant on $[a,c]$. Otherwise, let $[a_1,b_1]$ denote the right half $[c,b]$ of our interval. In either case, f is ε-unpleasant on $[a_1,b_1]$.

<u>General</u> <u>step.</u> Assume that $[a_n,b_n]$ is an interval contained in $[a,b]$ and that f is ε-unpleasant on $[a_n,b_n]$. Let c_n be the midpoint of $[a_n,b_n]$. As before, let $[a_{n+1},b_{n+1}]$ denote the left half $[a_n,c_n]$ of $[a_n,b_n]$ if f is ε-unpleasant on this half; otherwise, let $[a_{n+1},b_{n+1}]$ denote the right half. In either case, f is ε-unpleasant on $[a_{n+1},b_{n+1}]$.

We now have defined a sequence of intervals

$$[a_1,b_1],\ [a_2,b_2],\ldots$$

that is "nested" in the sense that each interval contains all its successors. Furthermore, each interval has half the length of the preceding one, and f is ε-unpleasant on each of them. It follows by an easy induction proof that the length of the n^{th} interval is

$$b_n - a_n = (b-a)/2^n.$$

Because the intervals are nested, we have

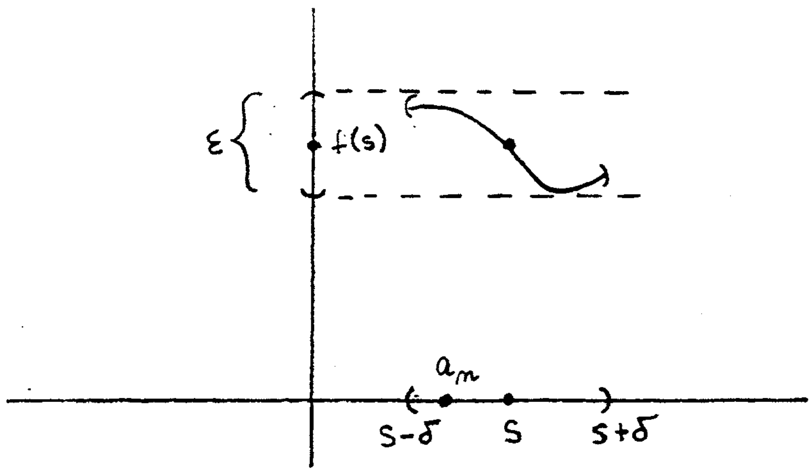
$$a \leq a_1 \leq a_2 \leq \ \ldots \ \leq b_2 \leq b_1 \leq b \ .$$

Let s be the least upper bound of the numbers a_i. Since all the numbers a_i belong to the interval $[a,b]$, so does s. Now we derive a contradiction. We have three cases, according as $a < s < b$, or $s = a$, or $s = b$.

Consider first the case where $a < s < b$. Since f is continuous at s, we may choose a neighborhood $(s-\delta, s+\delta)$ of s such that

$$|f(x) - f(s)| < \varepsilon/2$$

for all x in this neighborhood of s.



Because s is the least upper bound of the numbers a_i, there must be an n such that

$$s - \delta < a_n \leq s.$$

Because the a_i are increasing, we have

$$s - \delta < a_n \leq a_{n+1} \leq a_{n+2} \cdots \leq s.$$

Now let us choose m so large that $m > n$, and so that

$$(b - a)/2^m < \delta.$$

Then $b_m - a_m < \delta$, so that $b_m < a_m + \delta \leq s + \delta$. Then the interval $[a_m, b_m]$ is contained in the interval $(s - \delta, s + \delta)$. Therefore the span of f in the interval $[a_m, b_m]$ is at most ε.

It follows that f is ε-pleasant on $[a_m, b_m]$. Indeed, for __any__ partition of $[a_m, b_m]$, the span of f in each subinterval of the partition will be at most ε.

The other two cases proceed similarly. For example, suppose $s = a$. In this case, we have $a_n = a$ for all n. (This means merely that we chose the left half of the interval at each step of the construction.) Since f is continuous at a, there must be a δ such that

$$|f(x) - f(a)| < \varepsilon/2$$

for $a \leq x < a + \delta$. Choose m so large that $(b-a)/2^m < \delta$. Then $b_m - a_m < \delta$, so that $b_m < a_m + \delta = a + \delta$. Then the interval $[a_m, b_m]$ is contained in the interval $[a, a + \delta)$, so the span of f in the interval $[a_m, b_m]$ is at most ε. It follows that f is ε-pleasant in $[a_m, b_m]$, as before.

The proof when $s = b$ is similar. □

Here is an important application of this theorem:

__Theorem.__ __If__ f __is continuous on__ $[a,b]$, __then__ f __is bounded on__ $[a,b]$ __and integrable on__ $[a,b]$.

__Proof.__ Given $\varepsilon > 0$, choose a partition

$$x_0 < x_1 < \ldots < x_n$$

of $[a,b]$ such that the span of f on each closed subinterval of the partition is at most ε. Define

$$s_k = \inf \{f(x) \quad \text{for} \quad x_{k-1} \leq x \leq x_k\},$$

$$t_k = \sup \{f(x) \quad \text{for} \quad x_{k-1} \leq x \leq x_k\}.$$

Now f is bounded on $[a,b]$; indeed, $f(x)$ is at most the largest of the numbers $t_1, \ldots, t_n$ and at least the smallest of the numbers $s_1, \ldots, s_n$.

We define step functions s and t by letting their values equal s_k and t_k, respectively, on (x_{k-1}, x_k); at the partition points, we set $s(x_k) = t(x_k) = f(x_k)$. Then $s(x) \leq f(x) \leq t(x)$ for all x. Now $t_k - s_k \leq \varepsilon$ because the span of f on $[x_{k-1}, x_k]$ is at most ε. Therefore

$$\int_a^b t - \int_a^b s = \int_a^b (t-s) \leq \varepsilon(b-a).$$

The Riemann condition applies to show that f is integrable on $[a,b]$. □

Finally, we prove the third big theorem about continuous functions.

Theorem. (Extreme-value theorem). Let f be continuous on the closed interval $[a,b]$. Then there are points x_0 and x_1 of $[a,b]$ such that for every x in $[a,b]$, we have

$$f(x_0) \leq f(x) \leq f(x_1).$$

The number $M = f(x_1)$ is called the maximum-value of f on $[a,b]$, and the number $m = f(x_0)$ is called the minimum-value of $f(x)$ on $[a,b]$. Both are called extreme values of f on $[a,b]$.

Proof. We show that the point x_1 exists; the proof that x_0 exists is similar.

We know that f is bounded on $[a,b]$, by the previous theorem; define

$$M = \sup \{ f(x) \text{ for } x \text{ in } [a,b] \} .$$

We wish to show that $M = f(x_1)$ for some point x_1 of $[a,b]$. Suppose this is not true. We derive a contradiction.

Then by assumption, we have $f(x) < M$ for all x in $[a,b]$. Consider the function

$$g(x) = \frac{1}{M - f(x)} .$$

Since the denominator does not vanish, g is well-defined; because f is continuous, so is g. Therefore g is bounded on $[a,b]$, by the preceding theorem. Choose C so that $g(x) \leq C$ for x in $[a,b]$. Then

$$0 < \frac{1}{M - f(x)} \leq C,$$

so that $1/C \leq M - f(x)$, or $f(x) \leq M - 1/C$, for every x in $[a,b]$. This contradicts the fact that M is the least upper bound of the values of $f(x)$ on $[a,b]$. □

We shall use the extreme-value theorem shortly, when we prove the fundamental theorems of calculus.

Exercises on the intermediate-value, extreme-value, and small-span theorems.

1. Let $f(x) = x + [x]$ for $0 \leq x \leq 3$.

   (a) Draw the graph of f; show f is strictly increasing.

   (b) Define a function g by the following rule: If $y = f(x)$ for some x in $[0,3]$, let $g(y)$ equal that x. Because f is strictly increasing, g is well-defined. What is the domain of g?

2. Let $f(x) = x^4 + 2x^2 + 1$ for $0 \leq x \leq 10$.

   (a) Show f is strictly increasing; what is the domain of its inverse function g?

   (b) Find an expression for g, using radicals. ("Radicals" are the symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc.)

3. Let $f(x) = 2x^5 - 5x^4 + 5$ for $x \geq 2$. We will show later that f is strictly increasing (since its derivative is positive for $x > 2$).

   (a) Show that f is unbounded. (Hint: $f(x) > x^4(2x-5) > 2x - 5$.)

   (b) What is the domain of its inverse function g?

   [Note: A famous theorem of Modern Algebra states that it is not possible to express g in terms of algebraic operations and radicals.]

4. Let $f(x)$ be defined and continuous and strictly increasing for $x \geq 0$; suppose that $f(0) = a$.

   (a) Show that if f is unbounded, then f takes on every value greater than a.

   (b) If f is bounded, let M be the least upper bound of the values of f. Show that f takes on every value between a and M, but does not take on the value M.

5. Show by example that the conclusion of the intermediate value theorem can fail if f is only continuous on $[a,b)$ and bounded on $[a,b]$.

6. Show by example that the conclusion of the extreme value theorem can fail if f is only continuous on $[a,b)$ and bounded on $[a,b]$.

7. Let $f(x) = x$ for $0 \leq x < 1$; let $f(1) = 5$. Show that the conclusion of the small span theorem fails for the function $f(x)$.

8. A function f defined on $[a,b]$ is said to be <u>piecewise-continuous</u> if it is continuous except at finitely many points of $[a,b]$. Said differently, f is piecewise-continuous if there is some partition of $[a,b]$ such that f is continuous on each open subinterval determined by the partition.

   (a) Show that if f is bounded on $[a,b]$ and piecewise-continuous on $[a,b]$, then f is integrable on $[a,b]$. [Hint: Examine the proof we gave for piecewise-monotonic functions.]

   (b) Show that f can be piecewise-continuous on $[a,b]$ without being bounded on $[a,b]$.

9. Consider the function

$$f(x) = \begin{cases} (-1)^{[1/x]} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is integrable on $[0,1]$. Show that f is neither piecewise-monotonic nor piecewise-continuous on $[0,1]$.

10. <u>Challenge exercise.</u> Define a function f on the interval $[0,1]$ by setting $f(x) = 0$ if x is irrational ; and $f(x) = 1/n$ if x is a rational number of the form $x = m/n$, where m and n are positive integers having no common factors other than 1 ; and $f(0) = 1$.

(a) Show that f is integrable on $[0,1]$.

(b) Show that f is continuous at each irrational and discontinuous at each rational.

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