$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

In order to compute specific formulas for the derivatives of $\sin(x)$ and $\cos(x)$, we needed to understand the behavior of $\sin(x)/x$ near $x = 0$ (property B). In his lecture, Professor Jerison uses the definition of $\sin(\theta)$ as the y-coordinate of a point on the unit circle to prove that $\lim_{\theta \to 0}(\sin(\theta)/\theta) = 1$.

We switch from using x to using θ because we want to start thinking about the sine function as describing a ratio of sides in the triangle shown in Figure 1. The variable we're interested in is an angle, not a horizontal position, so we discuss $\sin(\theta)/\theta$ rather than $\sin(x)/x$.

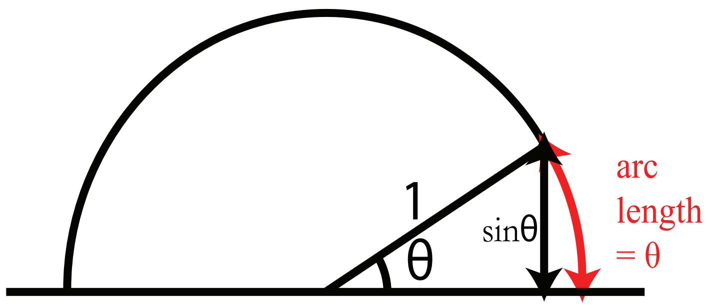

Figure 1: A circle of radius 1 with an arc of angle θ.

Our argument depends on the fact that when the radius of the circle shown in Figure 1 is 1, θ is the length of the highlighted arc. This is true when the angle θ is described in radians but NOT when it is measured in degrees.

Also, since the radius of the circle is 1, $\sin(\theta) = \frac{|\text{opposite}|}{|\text{hypotenuse}|}$ equals the length of the edge indicated (the hypotenuse has length 1).

In other words, $\sin(\theta)/\theta$ is the ratio of edge length to arc length. When $\theta = \pi/2$ rad, $\sin(\theta) = 1$ and $\sin(\theta)/\theta = 2/\pi \cong 2/3$. When $\theta = \pi/4$ rad, $\sin(\theta) = \sqrt{2}/2$ and $\sin(\theta)/\theta = 2\sqrt{2}/\pi \cong 9/10$. What will happen to the value of $\sin(\theta)/\theta$ as the value of θ gets closer and closer to 0 radians?

We see from Figure 2 that as θ shrinks, the length $\sin(\theta)$ of the segment gets closer and closer to the length θ of the curved arc. We conclude that as $\theta \to 0$,

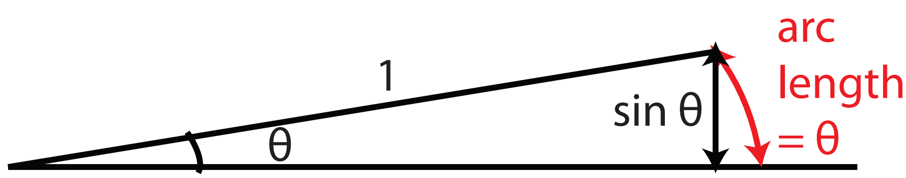

Figure 2: The sector in Fig. 1 as θ becomes very small

$\dfrac{\sin\theta}{\theta} \to 1$. In other words,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

This technique of comparing very short segments of curves to straight line segments is a powerful and important one in calculus; it is used several times in this lecture.