The Power Rule

What is the derivative of $\frac{d}{dx}x^r$? We answered this question first for positive integer values of r, for all integers, and then for rational values of r:

$$\frac{d}{dx}x^r = rx^{r-1}$$

We'll now prove that this is true for any *real* number r. We can do this two ways:

1st method: base e

Since $x = e^{\ln x}$, we can say:

$$\begin{aligned} x^r &= \left(e^{\ln x}\right)^r \\ x^r &= e^{r \ln x} \end{aligned}$$

We take the derivative of both sides to get:

$$\begin{aligned} \frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\frac{d}{dx}(r\ln x) \quad \text{(by the chain rule)} \\ &= e^{r\ln x}\left(\frac{r}{x}\right) \quad \text{(remember } r \text{ is constant)} \\ &= x^r\left(\frac{r}{x}\right) \quad \text{(because } x^r = e^{r\ln x}\text{)} \\ \frac{d}{dx}x^r &= rx^{r-1} \end{aligned}$$

2nd method: logarithmic differentiation

We define $f(x) = x^r$, and take the natural log of both sides to get $\ln f = r\ln x$. The technique of logarithmic differentiation requires us to we plug our function into the formula:

$$(\ln f)' = \frac{f'}{f}$$

So we first compute:

$$\begin{aligned} \ln f &= \ln x^r \\ \ln f &= r\ln x \end{aligned}$$

And then take the derivative of both sides to get:

$$(\ln f)' = \frac{r}{x}$$

Since $(\ln f)' = \frac{f'}{f}$, we have:

$$f' = f(\ln f') \quad = \quad x^r \left(\frac{r}{x}\right) = rx^{r-1}.$$

Look over the two methods again – the calculations are almost the same. This is typical. To use the second method we had to introduce a new symbol like u or f. In the first method we had to deal with exponents. It's worthwhile know both methods.