Evaluating an Interesting Limit

Using $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, calculate:

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$
2. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$
3. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n}$

Solution

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$

   The key to all of these problems is forcing them into the form $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. In this problem we do this by using rules of exponents to remove the 3 from the exponent.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot 3}\\ &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^3\\ &= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^3\\ &= e^3\end{aligned}$$

   How do we know that $\lim_{n\to\infty}(f(n)^3) = (\lim_{n\to\infty} f(n))^3$? This works because the function $g(x) = x^3$ is continuous; we could also justify it using what we know about limits of products.

2. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$

   In this problem we could easily remove the 5 from the exponent but there's no easy way to remove the numerator of 2. We must apply a change of variables to rewrite $\frac{2}{n}$ in the form $\frac{1}{m}$.

$$\begin{aligned}\frac{2}{n} &= \frac{1}{m}\\ 2 &= \frac{n}{m}\end{aligned}$$

$$n = 2m$$

Note that $\lim_{n\to\infty} n = \lim_{m\to\infty} 2m = \infty$. (Does it matter that m goes to infinity half as fast as n does? Why or why not?)

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n} &= \lim_{m\to\infty}\left(1+\frac{2}{2m}\right)^{5\cdot 2m} \\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{10m} \\
&= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{m}\right]^{10} \\
&= e^{10}
\end{aligned}$$

3. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n}$

This problem is very similar to the previous one.

$$\begin{aligned}
\frac{1}{2n} &= \frac{1}{m} \\
m &= 2n \\
n &= \frac{m}{2}
\end{aligned}$$

Again, $\lim_{n\to\infty} n = \lim_{m\to\infty} \frac{m}{2} = \infty$.

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n} &= \lim_{n\to\infty}\left(1+\frac{1}{2(\frac{m}{2})}\right)^{5\frac{m}{2}} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{\frac{5}{2}m} \\
&= e^{5/2}
\end{aligned}$$