Solving an Optimization Problem using Implicit Differentiation

Suppose you wish to build a grain silo with volume V made up of a steel cylinder and a hemispherical roof. The steel sheets covering the surface of the silo are quite expensive, so you wish to minimize the surface area of your silo. What height and radius should the silo have for a given volume V?

Although it is possible to solve this problem by the same method used in the can design question, it turns out to be much simpler to use implicit differentiation to find $\frac{d}{dr}SA$.

Answer this question for:

a) a silo with a circular floor (to keep out gophers) and

b) a silo with no built-in flooring (for use in regions with no gophers).

Solution

Our first task is to draw a sketch of the silo; this will help us in finding formulas for the total surface area and volume. The silo will have some radius r, and its cylindrical wall will have some height h.

a) A silo including a circular floor:

The surface area of the silo's cylindrical wall is the circumference of the base times the height, or $2\pi rh$; the surface area of the roof will be $\frac{1}{2}(4\pi r^2)$ and the floor will have surface area πr^2. Hence, the surface area of a silo with a floor is given by:

$$SA = 2\pi rh + 2\pi r^2 + \pi r^2 = 2\pi rh + 3\pi r^2.$$

The volume of the cylinder will be $\pi r^2 h$ and the volume of the hemispherical "attic" will be $\frac{1}{2}\left(\frac{4}{3}\pi r^2\right)$. The total volume of the silo will be:

$$V = \pi r^2 h + \frac{2}{3}\pi r^3.$$

We could use this equation to determine that $h = \frac{2r}{3} - \frac{V}{\pi r^2}$ and then substitute this in to the expression for surface area, but the resulting equation $\frac{d}{dr}SA = 0$ is unpleasant. Instead we apply implicit differentiation to the formula for surface area.

$$\begin{aligned} SA &= 2\pi rh + 3\pi r^2 \\ \frac{d}{dr}SA &= 2\pi h + 2\pi r\frac{dh}{dr} + 6\pi r \end{aligned}$$

We next use implicit differentiation on the formula for volume (or solve for h and differentiate normally) to find:

$$\begin{aligned}
V &= \pi r^2 h + \frac{2}{3}\pi r^3 \\
\frac{dV}{dr} &= \frac{dh}{dr}\pi r^2 + 2\pi r h + 2\pi r^2 \\
0 &= \frac{dh}{dr}\pi r^2 + 2\pi r h + 2\pi r^2 \\
-\pi r^2 \frac{dh}{dr} &= 2\pi r h + 2\pi r^2 \\
\frac{dh}{dr} &= -2 \cdot \frac{h+r}{r}.
\end{aligned}$$

Plugging this in to the formula for $\frac{d}{dr}SA$, we get:

$$\begin{aligned}
\frac{d}{dr}SA &= 2\pi h + 2\pi r \frac{dh}{dr} + 6\pi r \\
&= 2\pi h + 2\pi r \left(-2 \cdot \frac{h+r}{r}\right) + 6\pi r \\
&= 2\pi h - 4\pi\left(h+r\right) + 6\pi r \\
&= -2\pi h + 2\pi r.
\end{aligned}$$

We see that $\frac{d}{dr}SA = 0$ exactly when $h = r = \sqrt[3]{\frac{3V}{5\pi}}$.

Is this a minimum? There are no other critical points, and the extreme outcomes are a tall, thin silo with an extremely large surface area or a short, wide silo (as the height approaches 0 and the radius approaches $\sqrt[3]{\frac{3V}{2\pi}}$).

After using the volume formula to find expressions for r in terms of V, we see that in the case $h = 0$,

$$\begin{aligned}
SA &= 2\pi r h + 3\pi r^2 \\
&= 3\pi\left(\sqrt[3]{\frac{3V}{2\pi}}\right)^2 \\
SA &= \sqrt[3]{\frac{243V^2\pi}{4}}.
\end{aligned}$$

When $h = r = \sqrt[3]{\frac{3V}{5\pi}}$,

$$\begin{aligned}
SA &= 2\pi r h + 3\pi r^2 \\
&= 5\pi r^2 \\
&= 5\pi\left(\sqrt[3]{\frac{3V}{5\pi}}\right)^2 \\
&= \sqrt[3]{9V^2\pi}.
\end{aligned}$$

The solution in which $r = h$ has the least surface area.

Note that the silhouette of this silo is a rectangle with a semi-circle on top, which is closer to the shape of a water or oil tank than that of a silo. The tall, thin silos actually used to store grain take up less land area, leaving more room for farmland or other silos.

b) A silo with an open floor:

The formula for the volume of this silo is identical to the formula for the volume of one with floor included.

$$V = \pi r^2 h + \frac{2}{3}\pi r^3$$

The surface area of the silo is given by adding the surface area of the wall, which is $2\pi rh$, to the surface area of the roof, which is $2\pi r^2$.

$$SA = 2\pi rh + 2\pi r^2$$

For practice, we'll again use implicit differentiation to find $\frac{d}{dr}SA$:

$$\begin{aligned} SA &= 2\pi rh + 2\pi r^2 \\ \frac{d}{dr}SA &= 2\pi h + 2\pi r\frac{dh}{dr} + 4\pi r \end{aligned}$$

The calculation of $\frac{dh}{dr}$ using implicit differentiation is identical to the calculation in part (a):

$$\begin{aligned} V &= \pi r^2 h + \frac{2}{3}\pi r^3 \\ \frac{dV}{dr} &= \frac{dh}{dr}\pi r^2 + 2\pi rh + 2\pi r^2 \\ 0 &= \frac{dh}{dr}\pi r^2 + 2\pi rh + 2\pi r^2 \\ -\pi r^2\frac{dh}{dr} &= 2\pi rh + 2\pi r^2 \\ \frac{dh}{dr} &= -2 \cdot \frac{h+r}{r}. \end{aligned}$$

Plugging in, we find:

$$\begin{aligned} \frac{d}{dr}SA &= 2\pi h + 2\pi r\frac{dh}{dr} + 4\pi r \\ &= 2\pi h + 2\pi r\left(-2 \cdot \frac{h+r}{r}\right) + 4\pi r \\ &= 2\pi h - 4\pi\left(h+r\right) + 4\pi r \\ &= -2\pi h. \end{aligned}$$

In this case the extreme solution in which $h = 0$ is actually the minimal surface area solution. We conclude that the minimum surface area silo containing a given volume V is just a hemisphere, if the silo is built without a floor!