Average with Respect to Arc length

Find the average height of a point on a unit semicircle *with respect to the arc length θ.*

When taking averages, it's extremely important to specify the variable with respect to which the average is taking place. The answer may be different depending on the variable!
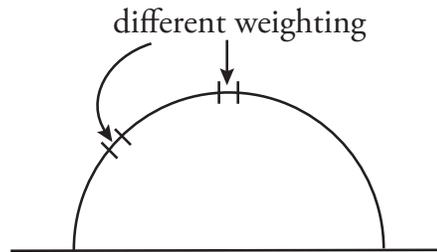

Figure 1: Equal arc lengths correspond to different distances on the x-axis.

As you can see from Figure 1, equal distances along the arc of the semicircle overshadow different lengths on the x-axis. Taking the average with respect to θ will weight the lower parts of the semicircle more heavily than the higher ones. We expect the average with respect to arc length be less than $\frac{\pi}{4}$.

Then average is still given by $\frac{1}{b-a} \int_a^b f(\theta)d\theta$. This time, $a = 0$ and $b = \pi$. The integrand is $y = \sin \theta$, which is the height of the semicircle in terms of θ. So our average height is:

$$
\begin{aligned}
\frac{1}{b-a} \int_a^b f(\theta)\,d\theta &= \frac{1}{\pi} \int_0^\pi \sin \theta \, d\theta \\
&= \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi \\
&= \frac{1}{\pi}\left(-\cos\pi - (-\cos 0)\right) \\
&= \frac{2}{\pi}
\end{aligned}
$$

Let's see if we can check our work. The average height with respect to arc length is $\frac{2}{\pi}$. The average height with respect to horizontal distance is $\frac{\pi}{4}$. Points

on the semicircle with lower height "count for more" in the computation with respect to arc length, so we'd expect a lower average. This does turn out to be the case because:

$$\frac{2}{\pi} < \frac{\pi}{4} \quad \text{if and only if} \quad 8 < \pi^2.$$

Since $\pi^2 > 9$ we have reason to believe we've found the right answer

Question: How do we interpret the result of the average with respect to arc length?

Answer: One way of thinking of it anticipates our next subject, which is probability. Suppose you picked a point at random along the base of the semicircle (with equal likelihood between -1 and 1) and checked the height above that point. The expected value of that height is given by the first calculation: $\frac{\pi}{4}$.

The second calculation tells you the expected value of the height of a point picked at random on the semicircle, if you were equally likely to pick any point on the semicircle.

Those two average heights are different because distance along the semicircle is different from distance along the x-axis.

Question: Shouldn't the average with respect to arc length have a *bigger* value because the arc length is *longer*?

Answer: Averages never work that way; when we multiply by $\frac{1}{b-a}$ we are dividing by the total length.

However, the average of a constant is that same constant regardless of what variable we use because we compensate by dividing by $b - a$. The difference between an integral and an average is that we're dividing by that total.

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