An Alternate Solution

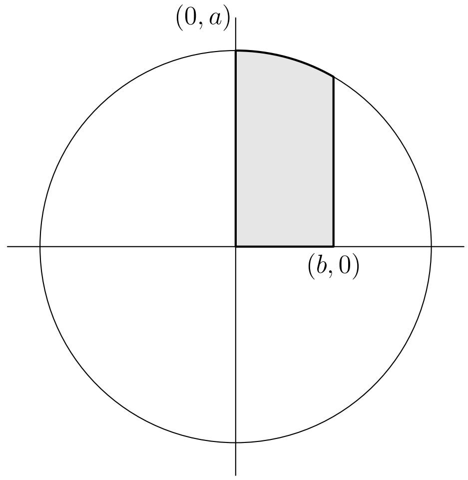

Figure 1: The area of the shaded region is $\int_0^b \sqrt{a^2 - x^2}\, dx$.

As Professor Miller explained in lecture, the area of the region shown in Figure 1 is $\int_0^b \sqrt{a^2 - x^2}\, dx$. Use the substitution $x = a\cos\theta$ to solve this integral. Hint: pay particular attention to your limits of integration.

Solution

In lecture, Professor Miller drew this picture in such a way that it was more natural to substitute $a \sin\theta$ rather than $a\cos\theta$. In this problem we verify that the two methods of finding the area yield the same result.

Notice that when $x = 0$, $\theta = \pi/2$ (assuming we're using polar coordinates). When $x = b$, $\theta = \arccos(\frac{b}{a}) < \pi/2$. As x increases, θ decreases. Although it's not strictly necessary, in the solution presented here we reverse the limits of integration to ensure that θ is increasing over the interval.

$$\begin{aligned}
\int_0^b \sqrt{a^2 - x^2}\, dx &= \int_{\frac{\pi}{2}}^{\arccos(\frac{b}{a})} \sqrt{a^2 - (a\cos\theta)^2} \cdot (-a\sin\theta)\, d\theta \\
&= -\int_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\cos^2\theta} \cdot (-a\sin\theta)\, d\theta \\
&= a\int_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} a\sqrt{(1 - \cos^2\theta)} \cdot \sin\theta\, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= a^2 \int_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} \sqrt{\sin^2\theta} \cdot \sin\theta \, d\theta \\
&= a^2 \int_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} \sin^2\theta \, d\theta \qquad \text{(from the diagram, } \sin\theta > 0\text{)} \\
&= a^2 \int_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= a^2 \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right]_{\arccos(\frac{b}{a})}^{\frac{\pi}{2}} \\
&= a^2 \left[\left(\frac{\frac{\pi}{2}}{2} - \frac{\sin(2 \cdot \frac{\pi}{2})}{4} \right) - \left(\frac{\arccos(\frac{b}{a})}{2} - \frac{\sin(2\arccos(\frac{b}{a}))}{4} \right) \right] \\
\int_0^b \sqrt{a^2 - x^2}\, dx &= a^2 \left(\frac{\pi}{4} - \frac{\arccos(\frac{b}{a})}{2} + \frac{\sin(2\arccos(\frac{b}{a}))}{4} \right)
\end{aligned}
$$

As Professor Miller noted, the identity $\sin(2t) = 2\sin t \cos t$ is helpful at this stage.

$$
\begin{aligned}
\int_0^b \sqrt{a^2 - x^2}\, dx &= a^2 \left(\frac{\pi}{4} - \frac{\arccos(\frac{b}{a})}{2} + \frac{2\sin(\arccos(\frac{b}{a}))\cos(\arccos(\frac{b}{a}))}{4} \right) \\
&= a^2 \left(\frac{\pi}{4} - \frac{\arccos(\frac{b}{a})}{2} + \frac{\sin(\arccos(\frac{b}{a}))(\frac{b}{a})}{2} \right)
\end{aligned}
$$

We need to understand $\arccos(\frac{b}{a})$ in order to further simplify this result. To do so, we can draw a diagonal line through the shaded region in Figure 1 to form a right triangle in which an leg of length b meets a hypotenuse of length a at an angle t. Then $\cos t = \frac{b}{a}$ or $t = \arccos(\frac{b}{a})$. Using this diagram and the Pythagorean theorem, we conclude that $\sin(\arccos(\frac{b}{a})) = \frac{\sqrt{a^2-b^2}}{a}$ and so:

$$
\begin{aligned}
\int_0^b \sqrt{a^2 - x^2}\, dx &= a^2 \left(\frac{\pi}{4} - \frac{\arccos(\frac{b}{a})}{2} + \frac{b}{2a} \frac{\sqrt{a^2 - b^2}}{a} \right) \\
&= a^2 \left(\frac{\pi}{4} - \frac{\arccos(\frac{b}{a})}{2} \right) + \frac{b}{2}\sqrt{a^2 - b^2} \\
\int_0^b \sqrt{a^2 - x^2}\, dx &= \frac{a^2(\frac{\pi}{2} - t)}{2} + \frac{b\sqrt{a^2 - b^2}}{2}
\end{aligned}
$$

When Professor Miller did this calculation by substituting $a\sin\theta$, he got the result $\int_0^b \sqrt{a^2 - y^2}\, dy = \frac{a^2\theta_0}{2} + \frac{b\sqrt{a^2-b^2}}{2}$. We can check our work by noting that our $\frac{\pi}{2} - t$ is equal to Professor Miller's θ_0.

Alternately, we could note that the area of the largest circular sector contained in the shaded region is $\frac{\pi}{4}a^2 - \frac{t}{2}a^2$, and the area of the remaining portion of the shaded region is $\frac{1}{2}b\sqrt{a^2 - b^2}$.