

Hi. Welcome back to recitation. You've been talking in class about partial fraction decomposition as a tool for integration. So remember that the point of partial fraction decomposition is that whenever you have a rational function, that is, one polynomial divided by another, that in principle, partial fraction decomposition lets you write any such expression as a sum of things, each of which is relatively easy to integrate. So the technique here is purely algebraic. And then you just apply integral rules that we've already learned.

So I have here four rational functions for you. And what I'd like you to do in each case, is try to decompose it into the general form that Professor Jerison taught you. So don't, I'm not asking you to-- if you'd like, you're certainly welcome to go ahead and compute the antiderivatives after you do that, but I'm not going to do it for you, or I'm not going to ask you to do it.

So what I'd like you to do, though, is for each of these four expressions, to break it out into the form of the partial fraction decomposition. So why don't you take a few minutes to do that, come back, and you can check your work against mine.

All right. Welcome back. Hopefully you had some fun working on these. They're a little bit tricky, I think, or I've picked them to be a little bit tricky. So let's go through them one by one. I guess I'll start with the first one.

So with the first one, I have $x^2 - 4x + 4$ over $x^2 - 8x$. Now, the first thing to do when you start the partial fraction method, is that you have to check that the degree of the numerator is smaller than the degree of the denominator. Now in this case, that's not true. The degree on top is 2, and the degree on bottom is 2. So we need to do long division. Or you know, we need to do something-- well, long division is the usual and always-- usual way, and always works-- in order to reduce the degree of the top here so that it's smaller than the degree at the bottom.

Now, I've done this ahead of time, and it's not too hard to check that this is equal to $1 + \frac{4x + 4}{x^2 - 8x}$. It's a relatively easy long division to do in any case.

So OK. So what we get after we do that process is we get something in front that's always a polynomial. And that's good, because remember, our goal is to, you know, manipulate this into a form where we can integrate it, and polynomials are easy to integrate. So then we usually just forget about this for the time being. And what we want to do is partial fraction decompose the second part.

So to do that, the first thing you need to do, once you've got a smaller degree on top than downstairs, is that you factor the denominator. So in this case, I'm going to keep the 1 plus, just so I can keep writing equals signs and be honest about it. But really, our focus now is just entirely on this second summand. So this is equal to $1 + \frac{4x + 4}{x^2 - 8x}$

4 is on top.

And OK, so we need to factor the denominator, if possible. And in this case, that's not only possible. It's pretty straightforward. We can factor out an x from both terms, and we see that the denominator is x times $x - 8$.

OK. Now, in this case, this is the simplest situation for partial fraction decomposition. We have the denominator is a product of distinct linear factors. So when the denominator is a product of distinct linear factors, what we get this is equal to, what the partial fraction theorem tells us we can write, is that this is equal to something, some constant, over the first factor plus some constant over the second factor. Now, if you had, you know, three factors, then you'd have three of these terms, one for each factor, if they were distinct linear factors.

So great. OK. Now we can apply the cover up method that Professor Jerison taught us. So again, the 1 doesn't really matter here. You can just ignore it. So we have this is equal to this sum, and we want to find the values of A and B that make this true. And then once we have values of A and B that make this true, the resulting expression will be all set to integrate. Right? This will just be easy. It's just a polynomial, in fact, just a constant. This is going to give us a logarithm, and this is going to give us a logarithm.

So once we find these constants A and B , we're all set to integrate. So OK, so how does the cover up method work? Well, so you cover up one of the factors. In this case, x . And you cover up, on the other side, everything that doesn't have x in the denominator, and also x . And then you go back here, and you plug in the appropriate values, you know, the x minus whatever. So in this case, that's x equals 0. And so over here, you get 4 over minus 8. So that gives us A is equal to 4 divided by negative 8, which is minus a half.

And now we can do the same thing with the $x - 8$ part. So we cover up $x - 8$, we cover up everything that doesn't have an $x - 8$ in it. So we've got, on the right-hand side, we get B , and on the left hand side, we have to plug in 8 here. So we plug 4 times 8 plus 4, which is 32 plus-- it's 36, and divided by eight, so that's 36 divided by 8 is 9 divided by 2.

So once you've got these values of A and B -- OK. So our original expression, now we substitute in these values of A and B , then we know that this is a true equality, and then the integration of this expression is just reduced completely to the integration of this easy-to-integrate expression. OK? And so this is the partial fraction decomposition here, with, you know A equals minus a half, and B equals nine halves.

So that's part (a). Let's go on to part (b). I have to remind myself what part (b) is. OK. So part (b) is x^2 divided by $x + 2$ to the fourth.

So in this case, in the denominator, we still have only linear factors, but we have repeated factors. And I mean, this is actually a particularly simple example, where we have only one factor, but it's repeated four times. Right? Fourth power.

So when you have that situation, the partial fraction decomposition looks a little bit different. And so what you get is that you have, for every repeated power, so for each-- this one appears four times, so you get four summands on the right-hand side, with increasing powers of this linear factor. So with increasing powers of $x - 1$. So this is going to be A -- or sorry, $x + 1$. A over $x + 1$ plus B over $x + 1$ squared plus C over $x + 1$ cubed plus D over $x + 1$ to the fourth.

Now remember, even though the degree here goes up in these later summands, what stays on top is the same. It just stays a constant. If this were a quadratic factor, it would stay linear. The top doesn't increase in degree when do this.

And a good simple way to check that you've got the right-- you know, before you solve for the constants, make sure you've got the correct abstract decomposition-- is to count the number of these constants that you're looking for. It should always match the degree of the denominator over here. So in this case, the degree of the denominator is 4, and there are 4 constants. So we've-- so that's a good way to check that you set the problem up right.

OK. So now, what do we do in this case? Well, the cover up method works, but it only works to find the highest degree term. Right? So we cover up $x + 1$ to the fourth, and we cover up everything with a smaller power of $x + 1$, and then we plug in negative 1. Right? Because we need $x + 1$ to be 0. So OK, so over here we get negative 1 squared is 1, so we get, right away, that D is equal to 1.

OK. But that doesn't give us A , B , or C . We can't get A , B , or C by the cover up method.

Now, there are a couple different ways you can proceed at this point. One thing you can do, which is something that often works, is you could plug in values. Well, so this will always work. I shouldn't say often works. You can start plugging in other values for x . And as you plug in other values for x , what you'll see is that for every value you plug in, you'll get a linear equation relating your variables-- A , B , C , and whatever, in this case, that's all we've got left, A , B , and C -- to each other. And so if you plug in three different values of x , say, you'll get three different linear equations with three different variables, and then you can solve them. That's one thing you can do.

Another thing you could do is you can multiply through by $x + 1$ to the fourth. So if you do that, you'll have-- on the left, you'll just have x squared, and on the right you'll have-- well, you'll have A times $x + 1$ cubed plus B times $x + 1$ squared, plus C times $x + 1$, plus D . And we already know that x is equal to 1. OK?

And so then, for those two things to be equal, they're equal as polynomials, all their coefficients have to be equal. So you can just look at the coefficients in that resulting expression, and ask, you know, which coefficients-- sorry. You can set coefficients on the two sides equal. The two polynomials are equal, all of their corresponding coefficients are equal. So you could look, you know, at this side and you'll say, oh, OK. So the coefficient of x cubed has to be the same as whatever the coefficient of x cubed over here is, and the coefficient x squared has to be the same as the coefficient of x squared, and so on. So that's another way to proceed.

Yeah, all right. Those are really your two best options. I like to multiply through, personally. So OK. So if I were to do that, in this case, on the left-hand side I'd get x squared equals A times x plus 1 cubed plus B times x plus 1 squared plus C times x plus 1 plus D . Except we already know that D is equal to 1, so I'm just going to write plus 1.

So OK. So actually, let me say that there are other things you could do. Which is, you could rearrange things and simplify algebraically before plugging in values, or before comparing coefficients. So let me give you one example of each of those three possibilities.

So for example, one thing you can do, is you can look at the highest degree coefficient. So as Professor Jerison said, the easiest coefficients are usually the high-order terms and the low-order terms. So in this case, the high-order terms would be-- this is a third degree polynomial on the right-hand side, and it's a second degree polynomial on the left. So the highest-order term here, x cubed, just appears in this one place as coefficient A . Right? This is $A x$ cubed plus something times x squared plus blah blah blah. And over here, we have no x cubeds. So we have x cubeds here, but no x cubeds here. That means the coefficient of x cubed here has to be 0, so A has to be 0.

OK. So A has to be equal to 0, and that simplifies everything a little bit. So now we get x squared equals B times x plus 1 squared plus C times x plus 1 plus 1. Now let me show you what I mean about algebraic manipulation. This 1, if you wanted, you could always just subtract it over to the other side. Right? And so then you'll have, on the left you'll have x squared minus 1. And x squared minus 1, you can write as x minus 1 times x plus 1 equals B times x plus 1 squared plus C times x plus 1. And then you can divide out by an x plus 1 everywhere. It appears in all terms. So you get x minus 1 equals B times x plus 1 plus C .

And now what this does for you, is you sort of just reduced the degree everywhere. And actually, you could substitute x equals minus 1 again, if you wanted to, for example. And here, so for example, if you substitute x equals minus 1, that's the same idea as what you do in the cover up method. This B term will just die completely, and you'll be left with negative 2 on the left. So you get C -- I'm going to have to move over here, sorry. C is equal to negative 2, then.

And also, you can do the one thing that I haven't done so far, is this plugging in nice choices of values. So another nice choice of value for x that we haven't used is x equals 0. So if you plug in x equals 0, you'll get minus 1 equals B plus c . So minus 1 equals B plus C . And since we just found C , that means that B is equal to 1.

All right. So-- oh boy, I'm using a lot of space, aren't I. All right. So in this case, we've got our coefficients, A equals 0, D equals 1, B equals 1, C equals minus 2. And that gives us the partial fraction decomposition.

Let's go back over here then, and look at question (c). So for (c), the question is, what is the partial fraction decomposition of $2x$ plus 2 divided by the quantity $4x$ squared plus 1 squared?

This one's really easy. This one is done. This is already partial fraction decomposed. Right? When you have-- so here we have an irreducible quadratic in the denominator. You can't factor this any further than it's gone. It also occurs to a higher power. So when you partial fraction decompose something like this, you want something linear over $4x$ squared plus 1, plus something linear over $4x$ squared plus 1 squared.

But we already have that, right? The first linear part is 0, and the second is something linear over $4x$ squared plus 1 squared. So to integrate this, it's already in a pretty good form. Now, you're actually going to write, if you wanted to integrate this, you would split it into two pieces, one with the $2x$ and then one with the 2. And the first one, you would just be a usual u substitution, and the second one, we would want some sort of trigonometric substitution. But this one is already ready for methods we already should be comfortable with. OK. So C , that's easy. It's done already. I'll put a check mark there, because that makes me happy. OK.

And for-- all right, and so for this last one, I'm also not going to write this one out. But the thing to notice here is the way I wrote it-- and this was really mean of me. Right? I wrote it as x squared minus 1 quantity squared. So a natural instinct is to say, aha! It's a quadratic repeated factor in the denominator. Right? But that's just because I was mean and I wrote it this way. That's not actually what this is. This is not irreducible. This factors. You can rewrite-- let me come back over here. This is for question (d). So you can rewrite x squared minus 1 squared as x minus 1 squared times x plus 1 squared. You can factor this x squared minus 1.

So when you factor this x minus 1, what you see is-- this isn't a problem that has one irreducible quadratic factor appearing to the second power. What it has is two linear factors, each appearing to the second power. So the partial fraction decomposition in this problem will be something like A over x minus 1 plus B over x minus 1 squared plus C over x plus 1, plus D over x plus 1 squared. That's what you'll get when you apply partial fraction decomposition to this problem. And then you have to solve for the coefficients A , B , C , and D .

So I'm not going to write that out myself, but I cleverly did it before I came on camera, so I can tell you what the answer is, if you want to check your work. So here we have A is equal to 0, B is equal to 1, C is equal to 1, and D

is equal to minus 3. So that's for the-- I didn't write it over here. That's for this particular numerator that we started with back over here. So for this particular fraction, if you carry out the partial fraction decomposition, what you get is right here.

So OK, so those are, that was a few more examples of the partial fraction decomposition. I hope you enjoyed them, and I'm going to end there.