18.02 Practice Exam 2B Thursday, Mar. 15, 2006 1:05–1:55

Problem 1.

Let $f(x, y) = xy - x^3$.

a) Sketch the level curve of $f(x, y)$ passing through the origin. Indicate the sign of f in the regions delimited by the level curve.

b) The function f has a critical point at the origin. Use your sketch to determine the type of critical point.

c) Find the gradient of f at $P : (1, 1)$.

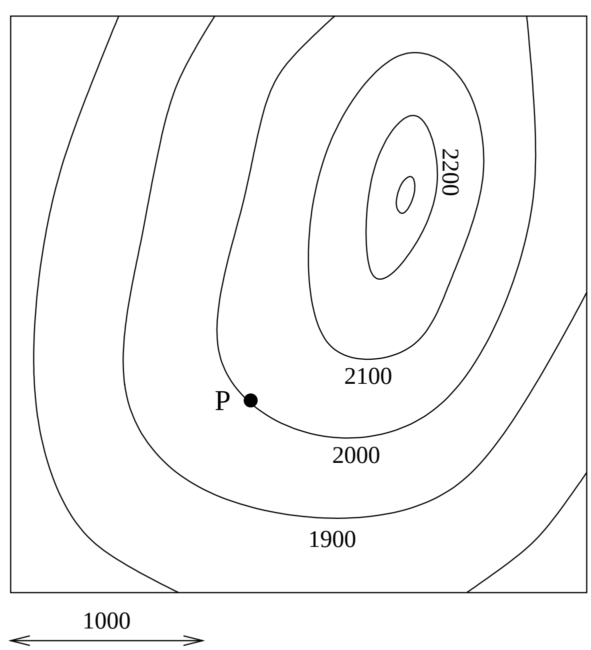
d) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

Problem 2. (15 points)

On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $\left(\frac{dh}{ds}\right)_{\hat{u}}$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

b) Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\frac{\partial h}{\partial y}$ at Q.



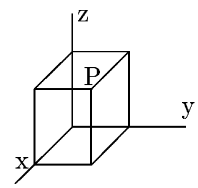
Problem 3. (10 points)

Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

Problem 4. (25 points: 5,5,5,10)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?

a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.



b) Find a critical point of f which lies in the first quadrant ($x > 0$, $y > 0$).

c) Determine the nature of this critical point by using the second derivative test.

d) Instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint. Write down the three equations involving the multiplier λ that one would need to solve.

Problem 5. (10 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

Problem 6. (15 points)

Suppose that $x^2y + xz^2 = 5$, and let $w = x^3y$. Express $\left(\frac{\partial w}{\partial z}\right)_y$ as a function of x, y, z, and evaluate it numerically when $(x, y, z) = (1, 1, 2)$.