18.02 Practice Exam 4A

Problem 1.

Let R be the solid region defined by the inequalities

$$x^2 + y^2 + z^2 \leq a^2, \quad x \geq 0, \quad y \geq 0$$

(a) *(15)* Set up a triple integral in **cylindrical** coordinates which gives the volume of R. (Put in integrand and limits, but DO NOT EVALUATE.)

(b) *(15)* Find the formula in **spherical** coordinates which gives the average distance of points of R to the xz-plane.
(Put in integrand and limits, but DO NOT EVALUATE.)

Problem 2.

Let $\vec{\mathbf{F}}$ be the vector field $\langle axz, -1 - bz^2, x^2 - 2yz + 4\rangle$.

(a) *(10)* For what values of a and b will $\vec{\mathbf{F}}$ be a conservative field?

(b) *(10)* For these values of a and b find a potential function f for $\vec{\mathbf{F}}$.
Use a systematic method and show your work.

Problem 3.

Let $\vec{\mathbf{F}} = \langle xz, yz + x, xy\rangle$.

(a) *(10)* Find $\vec{\nabla} \times \vec{\mathbf{F}}$.

(b) *(15)* Let C be the simple closed curve (oriented counterclockwise when viewed from above) $x - y + 2z = 10$ whose projection onto the xy-plane is the circle $(x-1)^2 + y^2 = 1$.
By using Stokes' theorem, compute $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Problem 4.

(a) *(20)* Use the divergence theorem to compute the flux of $\vec{\mathbf{F}} = (1 + y^2)\hat{\jmath}$ out of the curved part of the half-cylinder bounded by $x^2 + y^2 = a^2$ $(y \geq 0)$, $z = 0$, $z = b$, and $y = 0$. **Justify your answer.**

(b) *(5)* Suppose that S is a closed surface that lies entirely in $y < 0$.
Is the **outward** flux of $\vec{\mathbf{F}} = (1 + y^2)\hat{\jmath}$ through S positive, negative, or zero? **Justify your answer.**

Solutions to Practice Exam 4A.

Problem 1.

(a) $V = \int_{-a}^{a}\int_{0}^{\pi/2}\int_{0}^{\sqrt{a^2-z^2}} r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$

(b) $V = (1/4)(\text{volume of ball}) = (1/4)(4\pi a^3/3) = \pi a^3/3$.

The distance to xz-plane is $|y| = \rho \sin\varphi \sin\theta$. (Note that $y \geq 0$ and hence $\sin\theta \geq 0$ in the range we are considering.) Thus the average is

$$\frac{3}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi}\int_0^{a}(\rho\sin\varphi\sin\theta)\rho^2\sin\varphi\ \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta = \frac{3}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi}\int_0^{a}\rho^3\sin^2\varphi\sin\theta\ \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$

Problem 2.

(a) $P_y = 0 = Q_x$ (compatible).
$P_z = ax = R_x = 2x \implies a = 2$.
$Q_z = -2bz = R_y - = -2z \implies b = 1$.
ANSWER: $a = 2$, $b = 1$.

(b) $f_x = 2xz \implies f = x^2z + g(y,z) \implies$
$f_y = g_y = -1 - z^2 \implies g = -y - yz^2 + h(z)$. Therefore, $f = x^2z - y - yz^2 + h(z)$.
$f_z = x^2 - 2yz + h'(z) = x^2 - 2yz + 4 \implies h' = 4 \implies h = 4z(+\text{const})$.
In all, $f = x^2z - y - yz^2 + 4z(+\text{const})$

Problem 3.

(a) $$\vec{\nabla}\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz+x & xy \end{vmatrix} = \hat{\boldsymbol{\imath}}(x-y) - \hat{\boldsymbol{\jmath}}(y-x) + \hat{\boldsymbol{k}}(1) =$$
$= \langle x-y, x-y, 1\rangle$

(b) $\vec{\mathbf{N}} = \langle 1,-1,2\rangle$, $\mathrm{d}\vec{\mathrm{S}} = \frac{1}{2}\langle 1,-1,2\rangle \mathrm{d}x\mathrm{d}y$,

$$\vec{\nabla}\times\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathrm{S}} = \langle x-y, x-y, 1\rangle\cdot\frac{1}{2}\langle 1,-1,2\rangle\mathrm{d}x\mathrm{d}y = \frac{1}{2}((x-y)-(x-y)+2)\mathrm{d}x\mathrm{d}y = \mathrm{d}x\mathrm{d}y$$

$$\oint_C \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} = \iint_S \vec{\nabla}\times\vec{\mathbf{F}}\cdot\mathrm{dS} = \iint_{(x-1)^2+y^2<1}\mathrm{d}x\mathrm{d}y = \pi$$

Problem 4.

(a) S_1: $y = 0$, $-a \leq x \leq a$, $0 \leq z \leq b$, $\mathrm{d}\vec{\mathrm{S}} = -\hat{\boldsymbol{\jmath}}\ \mathrm{d}x\mathrm{d}z$, and $\overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = -(1+y^2)\mathrm{d}x\mathrm{d}z = -\mathrm{d}x\mathrm{d}z$ because $y = 0$. Thus,

$$\iint_{S_1} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = -\int_0^b \int_{-a}^a \mathrm{d}x\mathrm{d}z = -2ab$$

The top S_2 $(z = b)$ and bottom S_3 $(z = 0)$ have normal $\pm\hat{\boldsymbol{k}}$ and $\overrightarrow{\mathbf{F}} \cdot \hat{\boldsymbol{k}} = 0$, so the flux through these surfaces is zero. Therefore,

$$\iint_{S_1+S_2+S_3+S_4} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = -2ab + U$$

Where U is our unknown flux through the curved portion. On the other hand, by the divergence theorem,

$$\iint_{S_1+S_2+S_3+S_4} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = \iiint_D \overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}}\ \mathrm{dV} = \iiint_D 2y\ \mathrm{dV} = \int_0^b \int_0^\pi \int_0^a (2r\sin\theta) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

and

$$\int_0^b \int_0^\pi \int_0^a (2r\sin\theta) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \int_0^b \mathrm{d}z \int_0^\pi \sin\theta\mathrm{d}\theta \int_0^a 2r^2\mathrm{d}r = b(2)(\frac{2}{3}a^3)$$

In all,

$$-2ab + U = \frac{4}{3}a^3 b$$

so that the flux out of the half-cylinder through the curved portion S_4 is

$$\int_{S_4} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = U = \frac{4}{3}a^3 b + 2ab$$

(b) Recall that $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}} = 2y$. Since S encloses a region W that is entirely in the region where $y < 0$, then

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = \iiint_W 2y\ \mathrm{dV} < 0$$

In other words, the flux is always negative out of such surfaces S.