4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1

a) All vectors in the field are identical; continuously differentiable everywhere.
b) The vector at P has its tail at P and head at the origin; field is cont. diff. everywhere.
c) All vectors have unit length and point radially outwards; cont. diff. except at $(0,0)$.
d) Vector at P has unit length, and the clockwise direction perpendicular to OP.

4A-2 a) $a\,\mathbf{i} + b\,\mathbf{j}$ b) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^2}$ c) $f'(r)\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r}$

4A-3 a) $\mathbf{i} + 2\mathbf{j}$ b) $-r(x\,\mathbf{i} + y\,\mathbf{j})$ c) $\dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{r^3}$ d) $f(x,y)(\mathbf{i} + \mathbf{j})$

4A-4 $k \cdot \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2}$

4B. Line Integrals in the Plane

4B-1

a) On C_1: $y = 0,\ dy = 0$; therefore $\displaystyle\int_{C_1} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}.$

On C_2: $y = 1 - x^2,\ dy = -2x\,dx$; $\displaystyle\int_{C_2} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} (2x^2 - 1)\,dx - 4x^2\,dx$

$$= \int_{-1}^{1} (-2x^2 - 1)\,dx = -\left[\frac{2}{3}x^3 + x\right]_{-1}^{1} = -\frac{4}{3} - 2 = -\frac{10}{3}.$$

b) C: use the parametrization $x = \cos t,\ y = \sin t$; then $dx = -\sin t\,dt,\ dy = \cos t\,dt$

$$\int_C xy\,dx - x^2\,dy = \int_{\pi/2}^{0} -\sin^2 t\cos t\,dt - \cos^2 t\cos t\,dt = -\int_{\pi/2}^{0} \cos t\,dt = -\sin t\Big]_{\pi/2}^{0} = 1.$$

c) $C = C_1 + C_2 + C_3$; $C_1 : x = dx = 0$; $C_2 :\ y = 1 - x$; $C_3 : y = dy = 0$

$$\int_C y\,dx - x\,dy = \int_{C_1} 0 + \int_0^1 (1-x)dx - x(-dx) + \int_{C_3} 0 = \int_0^1 dx = 1.$$

d) $C : x = 2\cos t,\ y = \sin t$; $dx = -2\sin t\,dt$ $\displaystyle\int_C y\,dx = \int_0^{2\pi} -2\sin^2 t\,dt = -2\pi.$

e) $C :\ x = t^2,\ y = t^3$; $dx = 2t\,dt,\ dy = 3t^2\,dt$

$$\int_C 6y\,dx + x\,dy = \int_1^2 6t^3(2t\,dt) + t^2(3t^2\,dt) = \int_1^2 (15t^4)\,dt = 3t^5\Big]_1^2 = 3\cdot 31.$$

f) $\displaystyle\int_C (x+y)dx + xy\,dy = \int_{C_1} 0 + \int_0^1 (x+2)dx = \frac{x^2}{2} + 2x\Big]_0^1 = \frac{5}{2}.$

4B-2 a) The field $\mathbf{F}$ points radially outward, the unit tangent $\mathbf{t}$ to the circle is always perpendicular to the radius; therefore $\mathbf{F}\cdot\mathbf{t} = 0$ and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = 0$

b) The field $\mathbf{F}$ is always tangent to the circle of radius a, in the clockwise direction, and of magnitude a. Therefore $\mathbf{F} = -a\mathbf{t}$, so that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = -\int_C a\,ds = -2\pi a^2$.

4B-3 a) maximum if C is in the direction of the field: $C = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

b) minimum if C is in the opposite direction to the field: $C = -\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

c) zero if C is perpendicular to the field: $C = \pm\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$

d) max $= \sqrt{2}$, min $= -\sqrt{2}$: by (a) and (b), for the max or min $\mathbf{F}$ and C have respectively the same or opposite constant direction, so $\int_C \mathbf{F}\cdot d\mathbf{r} = \pm|\mathbf{F}|\cdot|C| = \pm\sqrt{2}$.

4C. Gradient Fields and Exact Differentials

4C-1 a) $\mathbf{F} = \nabla f = 3x^2y\,\mathbf{i} + (x^3+3y^2)\,\mathbf{j}$

b) (i) Using y as parameter, C_1 is: $x = y^2,\ y = y;$ thus $dx = 2y\,dy$, and

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} 3(y^2)^2y\cdot 2y\,dy + [(y^2)^3+3y^2]\,dy = \int_{-1}^{1}(7y^6+3y^2)\,dy = (y^7+y^3)\Big]_{-1}^{1} = 4.$$

b) (ii) Using y as parameter, C_2 is: $x = 1,\ y = y;$ thus $dx = 0$, and

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}(1+3y^2)\,dy = (y+y^3)\Big]_{-1}^{1} = 4.$$

b) (iii) By the Fundamental Theorem of Calculus for line integrals,

$$\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A).$$

Here $A = (1,-1)$ and $B = (1,1)$, so that $\displaystyle\int_C \nabla f\cdot d\mathbf{r} = (1+1) - (-1-1) = 4.$

4C-2 a) $\mathbf{F} = \nabla f = (xye^{xy}+e^{xy})\,\mathbf{i} + (x^2e^{xy})\,\mathbf{j}$.

b) (i) Using x as parameter, C is: $x = x,\ y = 1/x$, so $dy = -dx/x^2$, and so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (e+e)\,dx + (x^2e)(-dx/x^2) = (2ex - ex)\Big]_1^0 = -e.$$

b) (ii) Using the F.T.C. for line integrals, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(0,\infty) = 0 - e = -e.$

4C-3 a) $\mathbf{F} = \nabla f = (\cos x\cos y)\,\mathbf{i} - (\sin x\sin y)\,\mathbf{j}$.

b) Since $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is path-independent, for any C connecting $A : (x_0, y_0)$ to $B : (x_1, y_1)$, we have by the F.T.C. for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin x_1\cos y_1 - \sin x_0\cos y_0$$

This difference on the right-hand side is maximized if $\sin x_1\cos y_1$ is maximized, and $\sin x_0\cos y_0$ is minimized. Since $|\sin x\cos y| = |\sin x||\cos y| \le 1$, the difference on the right hand side has a maximum of 2, attained when $\sin x_1\cos y_1 = 1$ and $\sin x_0\cos y_0 = -1$.

(For example, a C running from $(-\pi/2, 0)$ to $(\pi/2, 0)$ gives this maximum value.)

4C-5 a) $\mathbf{F}$ is a gradient field only if $M_y = N_x$, that is, if $2y = ay$, so $a = 2$.

By inspection, the potential function is $f(x,y) = xy^2 + x^2 + c$; you can check that $\mathbf{F} = \nabla f$.

b) The equation $M_y = N_x$ becomes $e^{x+y}(x+a) = xe^{x+y} + e^{x+y}$, which $= e^{x+y}(x+1)$. Therefore $a = 1$.

To find the potential function $f(x,y)$, using Method 2 we have

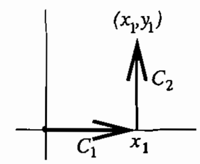
$f_x = e^y e^x(x+1) \;\Rightarrow\; f(x,y) = e^y x e^x + g(y)$.

Differentiating, and comparing the result with N, we find

$f_y = e^y x e^x + g'(y) = xe^{x+y}$; therefore $g'(y) = 0$, so $g(y) = c$ and $f(x,y) = x\,e^{x+y} + c$.

4C-6 a) $ydx - xdy$ is not exact, since $M_y = 1$ but $N_x = -1$.

b) $y(2x+y)\,dx + x(2y+x)\,dy$ is exact, since $M_y = 2x + 2y = N_x$.



Using Method 1 to find the potential function $f(x,y)$, we calculate the line integral over the standard broken line path shown, $C = C_1 + C_2$.

$$f(x_1,y_1) = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(x_1,y_1)} y(2x+y)\,dx + x(2y+x)\,dy.$$

On C_1 we have $y = 0$ and $dy = 0$, so $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$.

On C_2, we have $x = x_1$ and $dx = 0$, so $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{y_1} x_1(2y + x_1)\,dx = x_1 y_1^2 + x_1^2 y_1$.

Therefore, $f(x,y) = x^2y + xy^2$; to get all possible functions, add $+c$.

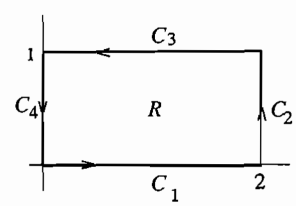
4D. Green's Theorem

4D-1 a) Evaluating the line integral first, we have $C:\ x = \cos t,\ y = \sin t,$ so

$$\oint_C 2y\,dx + x\,dy = \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 3\sin^2 t)\,dt = t - 3\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg]_0^{2\pi} = -\pi.$$

For the double integral over the circular region R inside the C, we have

$$\iint_R (N_x - M_y)\,dA = \iint_R (1-2)\,dA = -\text{ area of } R = -\pi.$$



b) Evaluating the line integral, over the indicated path $C = C_1 + C_2 + C_3 + C_4$,

$$\oint_C x^2dx + x^2dy = \int_0^2 x^2dx + \int_0^1 4\,dy + \int_2^0 x^2dx + \int_1^0 0\,dy \;=\; 4,$$

since the first and third integrals cancel, and the fourth is 0.

For the double integral over the rectangle R,

$$\iint_R 2x\,dA = \int_0^2\int_0^1 2x\,dydx = x^2\Big]_0^2 = 4.$$

c) Evaluating the line integral over $C = C_1 + C_2$, we have

$$C_1: \ x = x, \ y = x^2; \quad \int_{C_1} xy\,dx + y^2dy = \int_0^1 x\cdot x^2dx + x^4\cdot 2x\,dx = \frac{x^4}{4} + \frac{x^6}{3}\Big]_0^1 = \frac{7}{12}$$

$$C_2: \ x = x, \ y = x; \quad \int_{C_2} xy\,dx + y^2dy = \int_1^0 (x^2dx + x^2dx) = \frac{2}{3}x^3\Big]_1^0 = -\frac{2}{3}.$$

Therefore, $\displaystyle\oint_C xy\,dx + y^2dy = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}.$

Evaluating the double integral over the interior R of C, we have

$$\iint_R -x\,dA = \int_0^1\int_{x^2}^{x} -x\,dy dx;$$

evaluating: Inner: $\displaystyle -xy\Big]_{y=x^2}^{y=x} = -x^2 + x^3$; Outer: $\displaystyle -\frac{x^3}{3} + \frac{x^4}{4}\Big]_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}.$

4D-2 By Green's theorem, $\displaystyle\oint_C 4x^3y\,dx + x^4\,dy = \iint (4x^3 - 4x^3)\,dA = 0.$

This is true for every closed curve C in the plane, since M and N have continuous derivatives for all x, y.

4D-3 We use the symmetric form for the integrand since the parametrization of the curve does not favor x or y; this leads to the easiest calculation.

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt = \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\,dt$$

Using $\sin^2 t\cos^2 t = \frac{1}{4}(\sin 2t)^2 = \frac{1}{4}\cdot\frac{1}{2}(1 - \cos 4t)$, the above $\displaystyle = \frac{3}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)_0^{2\pi} = \frac{3\pi}{8}.$

4D-4 By Green's theorem, $\displaystyle\oint_C -y^3dx + x^3dy = \iint_R (3x^2 + 3y^2)\,dA > 0$, since the integrand is always positive outside the origin.

4D-5 Let C be a square, and R its interior. Using Green's theorem,

$$\oint_C xy^2dx + (x^2y + 2x)\,dy = \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = 2(\text{area of } R).$$

4E. Two-dimensional Flux

4E-1 The vector $\mathbf{F}$ is the velocity vector for a rotating disc; it is at each point tangent to the circle centered at the origin and passing through that point.

a) Since $\mathbf{F}$ is tangent to the circle, $\mathbf{F}\cdot\mathbf{n} = 0$ at every point on the circle, so the flux is 0.

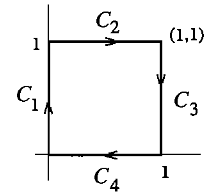
b) $\mathbf{F} = x\,\mathbf{j}$ at the point $(x,0)$ on the line. So if $x_0 > 0$, the flux at x_0 has the same magnitude as the flux at $-x_0$ but the opposite sign, so the net flux over the line is 0.

c) $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = x\,\mathbf{j}\cdot -\mathbf{j} = -x$. Thus $\displaystyle\int \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{1}{2}.$

4E-2 All the vectors of $\mathbf{F}$ have length $\sqrt{2}$ and point northeast. So the flux across a line segment C of length 1 will be

a) maximal, if C points northwest;
b) minimal, if C point southeast;
c) zero, if C points northeast or southwest;
d) -1, if C has the direction and magnitude of $\mathbf{i}$ or $-\mathbf{j}$; the corresponding normal vectors are then respectively $-\mathbf{j}$ and $-\mathbf{i}$, by convention, so that $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i}+\mathbf{j})\cdot -\mathbf{j} = -1.$ or $(\mathbf{i}+\mathbf{j})\cdot -\mathbf{i} = -1.$
e) respectively $\sqrt{2}$ and $-\sqrt{2}$, since the angle θ between $\mathbf{F}$ and n is respectively 0 and π, so that respectively $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}|\cos\theta = \pm\sqrt{2}$.

4E-3 $$\int_C M\,dy - N\,dx = \int_C x^2dy - xy\,dx = \int_0^1 (t+1)^2 2t\,dt - (t+1)t^2\,dt$$
$$= \int_0^1 (t^3+3t^2+2t)\,dt = \frac{t^4}{4}+t^3+t^2\Big]_0^1 = \frac{9}{4}.$$



4E-4 Taking the curve $C = C_1 + C_2 + C_3 + C_4$ as shown,
$$\int_C x\,dy - y\,dx = \int_{C_1} 0 + \int_0^1 -dx + \int_1^0 dy + \int_{C_4} 0 = -2.$$

4E-5 Since $\mathbf{F}$ and $\mathbf{n}$ both point radially outwards, $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = a^m$, at every point of the circle C of radius a centered at the origin.

a) The flux across C is $a^m \cdot 2\pi a = 2\pi a^{m+1}$.
b) The flux will be independent of a if $m = -1$.

4F. Green's Theorem in Normal Form

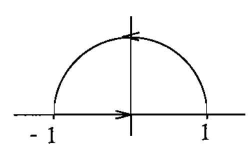
4F-1 a) both are 0 b) div $\mathbf{F} = 2x+2y$; curl $\mathbf{F} = 0$ c) div $\mathbf{F} = x+y$; curl $\mathbf{F} = y-x$

4F-2 a) div $\mathbf{F} = (-\omega y)_x + (\omega x)_y = 0$; curl $\mathbf{F} = (\omega x)_x - (-\omega y)_y = 2\omega$.

b) Since $\mathbf{F}$ is the velocity field of a fluid rotating with constant angular velocity (like a rigid disc), there are no sources or sinks: fluid is not being added to or subtracted from the flow at any point.

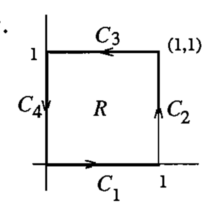
c) A paddlewheel placed at the origin will clearly spin with the same angular velocity ω as the rotating fluid, so by (15), the curl should be 2ω at the origin. It is less obvious that the curl is 2ω at all other points as well.

4F-3 The line integral for flux is $\int_C x\,dy - y\,dx$; its value is 0 on any segment of the x-axis since $y = dy = 0$; on the upper half of the unit semicircle (oriented counterclockwise), $\mathbf{F}\cdot\mathbf{n} = 1$, so the flux is the length of the semicircle: π.



Letting R be the region inside C, $\displaystyle\iint_R \text{div }\mathbf{F}\,dA = \iint_R 2\,dA = 2(\pi/2) = \pi.$

4F-4 For the flux integral $\displaystyle\oint_C x^2dy - xy\,dx$ over $C = C_1 + C_2 + C_3 + C_4$,



we get for the four sides respectively $\displaystyle\int_{C_1} 0 + \int_0^1 dy + \int_1^0 -x\,dx + \int_{C_4} 0 = \frac{3}{2}.$

For the double integral, $\iint_R \text{div } \mathbf{F}\, dA = \iint_R 3x\, dA = \int_0^1 \int_0^1 3x\, dy dx = \frac{3}{2}x^2\Big]_0^1 = \frac{3}{2}.$

4F-5 $r = (x^2+y^2)^{1/2} \Rightarrow r_x = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{r}$; by symmetry, $r_y = \frac{y}{r}$.

To calculate div $\mathbf{F}$, we have $M = r^n x$ and $N = r^n y$; therefore by the chain rule, and the above values for r_x and r_y, we have

$$M_x = r^n + nr^{n-1}x\cdot\frac{x}{r} = r^n + n\,r^{n-2}x^2; \quad \text{similarly (or by symmetry),}$$
$$N_y = r^n + nr^{n-1}y\cdot\frac{y}{r} = r^n + n\,r^{n-2}y^2, \quad \text{so that}$$
$$\text{div } \mathbf{F} = M_x + N_y = 2r^n + nr^{n-2}(x^2+y^2) = r^n(2+n), \quad \text{which } = 0 \text{ if } n = -2.$$

To calculate curl $\mathbf{F}$, we have by the chain rule

$$N_x = nr^{n-1}\cdot\frac{x}{r}\cdot y; \quad M_y = nr^{n-1}\cdot\frac{y}{r}\cdot x, \quad \text{so that curl } \mathbf{F} = N_x - M_y = 0, \text{ for all } n.$$

4G. Simply-connected Regions

4G-1 Hypotheses: the region R is simply connected, $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ has continuous derivatives in R, and curl $\mathbf{F} = 0$ in R.

Conclusion: $\mathbf{F}$ is a gradient field in R (or, $M\,dx + N\,dy$ is an exact differential).

a) curl $\mathbf{F} = 2y - 2y = 0$, and R is the whole xy-plane. Therefore $\mathbf{F} = \nabla f$ in the plane.

b) curl $\mathbf{F} = -y\sin x - x\sin y \neq 0$, so the differential is not exact.

c) curl $\mathbf{F} = 0$, but R is the exterior of the unit circle, which is not simply-connected; criterion fails.

d) curl $\mathbf{F} = 0$, and R is the interior of the unit circle, which is simply-connected, so the differential is exact.

e) curl $\mathbf{F} = 0$ and R is the first quadrant, which is simply-connected, so $\mathbf{F}$ is a gradient field.

4G-2 a) $f(x,y) = xy^2 + 2x$ b) $f(x,y) = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}$

c) Using Method 1, we take the origin as the starting point and use the straight line to (x_1,y_1) as the path C. In polar coordinates, $x_1 = r_1\cos\theta_1,\ y_1 = r_1\sin\theta_1$; we use r as the parameter, so the path is $C: x = r\cos\theta_1,\ y = r\sin\theta_1,\ 0 \le r \le r_1$. Then

$$f(x_1,y_1) = \int_C \frac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \int_0^{r_1} \frac{r\cos^2\theta_1 + r\sin^2\theta_1}{\sqrt{1-r^2}}\,dr$$
$$= \int_0^{r_1} \frac{r}{\sqrt{1-r^2}}dr = -\sqrt{1-r^2}\Big]_0^{r_1} = -\sqrt{1-{r_1}^2} + 1.$$

Therefore, $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$

Another approach: $x\,dx + y\,dy = \frac{1}{2}d(r^2)$; therefore $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \dfrac{1}{2}\dfrac{d(r^2)}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$
(Think of r^2 as a new variable u, and integrate.)

4G-3 By Example 3 in Notes V5, we know that $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\dfrac{1}{r}\right)$.

Therefore, $\displaystyle\int_{(1,1)}^{(3,4)} = -\frac{1}{r}\Big]_{\sqrt{2}}^{5} = \frac{1}{\sqrt{2}} - \frac{1}{5}.$

4G-4 By Green's theorem $\displaystyle\oint_C xy\,dx + x^2\,dy = \iint_R x\,dA.$

For any plane region of density 1, we have $\iint_R x\,dA = \bar{x}\cdot$(area of R), where $\bar{x}$ is the x-component of its center of mass. Since our region is symmetric with respect to the y-axis, its center of mass is on the y-axis, hence $\bar{x} = 0$ and so $\iint_R x\,dA = 0$.

4G-5

a) yes
b) no (a circle surrounding the line segment lies in R, but its interior does not)
c) yes (no finite curve could surround the entire positive x-axis)
d) no (the region does not consist of one connected piece)
e) yes if $\theta_0 < 2\pi$; no if $\theta_0 \geq 2\pi$, since then R is the plane with $(0,0)$ removed
f) no (a circle between the two boundary circles lies in R, but its interior does not)
g) yes

4G-6

a) continuously differentiable for $x, y > 0$; thus R is the first quadrant without the two axes, which is simply-connected.

b) continuous differentiable if $r < 1$; thus R is the interior of the unit circle, and is simply-connected.

c) continuously differentiable if $r > 1$; thus R is the exterior of the unit circle, and is not simply-connected.

d) continuously differentiable if $r \neq 0$; thus R is the plane with the origin removed, and is not simply-connected.

e) continuously differentiable if $r \neq 0$; same as (d).

4H. Multiply-connected Regions

4H-1 a) 0; 0 b) 2; 4π c) -1; -2π d) -2; -4π

4H-2 In each case, the winding number about each of the points is given, then the value of the line integral of $\mathbf{F}$ around the curve.

a) $(1, -1, 1)$; $2 - \sqrt{2} + \sqrt{3}$
b) $(-1, 0, 1)$; $-2 + \sqrt{3}$
c) $(-1, 0, 0)$; -2
d) $(-1, -2, 1)$; $-2 - 2\sqrt{2} + \sqrt{3}$