Rough Solutions for PSet 12

1. (12.10:7) We first parameterize the sphere of radius a by $r(\phi, \theta) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$ where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. (We write ϕ first so that the normal points outward.) Then the integrals become

$$\int\int_S xy\, dy \wedge\, dz = \int_0^{\pi}\int_0^{2\pi} a^2 \sin^2\phi\cos\theta\sin\theta(a^2\cos\theta\sin^2\phi)\, d\theta\, d\phi,$$

$$\int\int_S yz\, dz \wedge\, dx = \int_0^{\pi}\int_0^{2\pi} a^2 \sin\theta\sin\phi\cos\phi(a^2\sin\theta\sin^2\phi)\, d\theta\, d\phi,$$

$$\int\int_S x^2\, dx \wedge\, dy = \int_0^{\pi}\int_0^{2\pi} a^2 \cos^2\theta\sin^2\phi(a^2\cos\phi\sin\phi)\, d\theta\, d\phi.$$

From here the work is standard. Notice the second and third integral can be combined and simplified.

2. (12.10:12) We want to evaluate $\int\int_S F\cdot n dS$ and here $n = (x, y, z)$ since S is a hemisphere. Thus $F \cdot n = x^2 - 2xy - y^2 + z^2$. As with the problem above, we parameterize the hemisphere by $r(\phi, \theta)$ where $0 \le \phi \le \pi/2$ and $\theta \in [0, 2\pi]$. So the problem is to evaluate

$$\int_0^{\pi/2}\int_0^{2\pi} \left(\cos^2\theta\sin^2\phi - 2\cos\theta\sin\theta\sin^2\phi - \sin^2\theta\sin^2\phi + \cos^2\phi\right)\sin\phi\, d\theta\, d\phi.$$

The first three terms integrate to zero in θ so the only term that carries through to the second integral is $2\pi\int_0^{\pi/2}\cos^2\phi\sin\phi\, d\phi = 2\pi/3$.

3. (12.10:13) There's actually nothing to add in. This is because $F \cdot n$ on the disk is zero. ($z = 0$ on the x, y-plane and $n = (0, 0, -1)$). The book apparently has a different answer. So let's just check this by using the divergence theorem. That is

$$\int\int\int_V div(F)\, dx\, dy\, dz = \int\int\int_V 1\, dx\, dy\, dz = Vol(V)$$

where V is the upper solid hemisphere. Thus, $Vol(V) = 2\pi/3$. But since

$$\int\int\int_V div(F)\, dx\, dy\, dz = \int\int_{S_{cap}} F\cdot n\, dS_{cap} + \int\int_{S_{disk}} F\cdot n\, dS_{disk}$$

we see the second term can contribute nothing.

4. (12.13:3) The line integral of interest will be over the square with side length 2, with two sides on the x- and y-axes. Before we write out all of the details, notice that on the x, y-plane, $F(x, y, 0) = (y, 0, 0)$ so the only parts of the line integral that matter are the parts parallel to the x-axis (do $dx \neq 0$). Thus, the problem reduces to finding

$$\int_0^2 F(t, 0, 0) \cdot (1, 0, 0)\, dt - \int_0^2 F(t, 2, 0) \cdot (1, 0, 0)\, dt.$$

The first integral is zero since $F \equiv 0$ on that curve. The second becomes

$$-\int_0^2 2\, dt = -4.$$

5. (12.13:11) It is enough to show that $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$. Then we can use Stokes' Theorem:

$$\int_{\partial S} (\mathbf{a} \times \mathbf{r}) \cdot\ d\gamma = \int\int_S \nabla \times (\mathbf{a} \times \mathbf{r}) \cdot n\, dS.$$

But this is just a straightforward calculation. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{r} = (x, y, z)$. Then

$$\mathbf{a} \times \mathbf{r} = (a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x).$$

Taking

$$curl(\mathbf{a} \times \mathbf{r}) = (a_1 + a_1, a_2 + a_2, a_3 + a_3)$$

gives the result.