Solutions for PSet 3

1. First we note that $T(P)$ is again a parallelepiped. To see this notice that if $x \in T(P)$ then $x = T(\sum_{i=1}^{n} c_i v_i)$ where $0 \le c_i \le 1$. By linearity $x = \sum_i c_i T(v_i)$ for $0 \le c_i \le 1$. That is, $T(P) = \{x \in \mathbb{R}^n | x = \sum_i c_i T(v_i), 0 \le c_i \le 1\}$ and this is precisely the definition of a parallelepiped.

   Now, we determine the volume. If V denotes the matrix formed such that the ith row is v_i, then:
   $$Vol(P) = |det(V)|$$
   Let W denote matrix with $w_i = m(T)v_i$ as the ith row, then by the definition of $T(P)$ above:

   $$Vol(T(P)) = |det(W)| = |det(m(T)V)| = |det(m(T))det(V)| = |det(m(T))|vol(P).$$

2. (14.4:23) Since F is continuous, we can apply the first fundamental theorem of calculus to determine a derivative for the expression on the right hand side of the equation. Thus

   $$F'(x) = e^x A + xe^x A + \frac{1}{x}F(x) - \frac{1}{x^2}\int_1^x F(t)\,dt = 2e^x A + xe^x A \qquad \forall\, x > 0.$$

   Now we apply the second fundamental theorem of calculus and the above result to get:
   $$F(x) - F(1) = \int_1^x F'(t)\,dt = A\int_1^x (2e^t + te^t)\,dt$$
   Integrating by parts we have:

   $$F(x) - F(1) = A(2e^t + te^t - e^t)\Big|_1^x = A(e^t + te^t)\Big|_1^x = A(e^x + xe^x) - 2Ae$$

   Substituting $F(1) = eA$ we observe

   $$F(x) = Ae^x + Axe^x - Ae$$

3. First the vectors $(3,2,4) - (1,0,0) = (2,2,4)$ and $(1,-1,1) - (1,0,0) = (0,-1,1)$ are parallel to the plane P. Thus, the vector $(2,2,4) \times (0,-1,1) = (6,-2,-2)$ is orthogonal to the plane P. The unit normal vector is thus

$\frac{1}{\sqrt{11}}(3,-1,-1) = \mathbf{n}$. Now, we consider the projection of the point $(1,0,0)$ on the plane onto this normal direction $\mathbf{n}$:

$$proj_{\mathbf{n}}(1,0,0) = \frac{(1,0,0)\cdot\mathbf{n}}{||\mathbf{n}||^2}\mathbf{n} = \frac{3}{\sqrt{11}}\mathbf{n} = \frac{3}{11}(3,-1,-1)$$

This gives the nearest point of P to the origin.

4. (14.9:12)

   (a) The motion is counterclockwise. This can be seen as follows: For t such that $g(t) > 0$, the particle is at some point on the upper half of the ellipse. Here as the horizontal component of the velocity is negative (leftward), the upper portion of the ellipse is carved out from right to left. This is counterclockwise motion.

   (b) Since $r(t) = (f(t), g(t))$ gives the ellipse, we see

   $$3f(t)^2 + g(t)^2 = 1$$

   Taking the derivative of both sides with respect to t, and substituting the given fact that $f'(t) = -g(t)$ we see

   $$6f(t)f'(t)+2g(t)g'(t) = 0 \implies -6f(t)g(t)+2g(t)g'(t) = 0 \implies g(t)[g'(t)-3f(t)] = 0$$

   That is, for $g(t) \neq 0$ we have $g'(t) = 3f(t)$. Hence the vertical component of velocity is proportional to $f(t)$. When $g(t) = 0$, use the continuity of $g'(t)$ to get the same result.

   (c) Notice that the position vector (and therefore f and g) should be periodic. Moreover, $g'(t) = 3f(t)$ implies $g''(t) = 3f'(t) = -3g(t)$. Assuming we've parameterized so that $g(0) = 0$, we see $g(t) = \sin(\sqrt{3}t)$ solves the above second order differential equation. Since the period of this function is $\frac{2\pi}{\sqrt{3}}$, this is precisely the time needed to travel around the ellipse once.

5. (14.9:15)

   (a) As $\mathbf{r}'(t) = A \times \mathbf{r}(t)$, $\mathbf{a}(t) = \mathbf{r}''(t) = A \times \mathbf{r}'(t)$. Thus, $A \cdot \mathbf{a}(t) = A \cdot (A \times \mathbf{r}'(t)) = 0$ as per the defining properties of the cross-product. Thus $\mathbf{a}(t)$ is orthogonal to both A and $\mathbf{r}'(t)$.

(b) Recall that $\mathbf{a}(t) = v(t)T'(t) + v'(t)T(t)$. As shown in part (a), $\mathbf{a}(t)$ is orthogonal to $\mathbf{r}'(t)$ and therefore to the tangential vector $T(t)$. This means that the component $v'(t)$ must be identically zero. By the mean value theorem, this implies $v(t)$ is constant. Now since the speed is constant we would like to compute it at $t = 0$. While the speed is not defined at $t = 0$ (since the initial condition is for $t = 0$), the constancy of the function means we can take the limit as $t \to 0^+$ of this constant function to get its value:

$$v(t) = \lim_{t \to 0^+} \mathbf{v}(t) = \lim_{t \to 0^+} A \times \mathbf{r}(t) = A \times r(0) = A \times B$$

The speed is $|A \times B| = |A||B| \sin \theta$.

(c) The following steps enable us to determine the curve defining $\mathbf{r}(t)$:

$$\begin{aligned} \mathbf{r}'(t) &= A \times \mathbf{r}(t) \\ A \cdot \mathbf{r}'(t) &= A \cdot (A \times \mathbf{r}(t) = 0 \\ \implies \mathbf{r}(t) \cdot A & \quad \text{is constant} \end{aligned}$$

So the angle between the position vector and A is fixed at θ. Also from part(a), $\mathbf{a} \perp A$ implies the curve sits in a plane orthogonal to A. Both $\mathbf{r}'(t)$ and $\mathbf{a}(t)$ are orthogonal to A. This could imply $\mathbf{a}(t) \parallel \mathbf{r}'(t)$, but this is not possible as $\mathbf{a}(t) = \mathbf{r}''(t) \perp \mathbf{r}'(t)$. As acceleration and velocity are perpendicular, $\mathbf{r}(t)$ must define a circle. Precisely, $\mathbf{r}(t)$ is a circle in a plane orthogonal to A that makes a cone angle of θ with A.

6. (14.13:16) As the area of the function f is proportional to its arc length over the same interval, we can write:

$$A(x) \equiv \int_a^x f(t)dt = c \int_a^x \sqrt{1 + (f'(t))^2} dt$$

Note: To compute arc length we are assuming that f' is continuous.

Applying the second fundamental theorem of calculus we see

$$A'(x) = f(x) \text{ and } A'(x) = c\sqrt{1 + (f'(x))^2}$$

Or equivalently:

$$f(x) = c\sqrt{1 + (f'(x))^2} \implies \frac{f^2(x)}{c^2} - (f'(x))^2 = 1$$

Notice a first solution is if $f' \equiv 0$ then $f^2 = c^2$ works and f is just a constant function.

Now suppose $f' \neq 0$. In this case, the equation above is solved by:

$$f(x) = c\cosh(\frac{x}{c}) \text{ and } f'(x) = \sinh(\frac{x}{c})$$

As $\cosh^2(x) - \sinh^2(x) = 1$, the constraint of $f^2/c^2 - f'^2 = 1$ is fulfilled. A more complete derivation of this solution can be found in the final exam year 2010. Therein the Taylor series for two functions u, v such that $u' = v, v' = u$ with conditions $u(0) = 0, v(0) = 1$ are examined. It is shown that u and v were the functions $\cosh x$ and $\sinh x$, and follow the property $\cosh^2(x) - \sinh^2(x) = 1$.