

Matrices

We have already defined what we mean by a matrix. In this section, we introduce algebraic operations into the set of matrices.

Definition. If A and B are two matrices of the same size, say k by n , we define $A + B$ to be the k by n matrix obtained by adding the corresponding entries of A and B , and we define cA to be the matrix obtained from A by multiplying each entry of A by c . That is, if a_{ij} and b_{ij} are the entries of A and B , respectively, in row i and column j , then the entries of $A + B$ and of cA in row i and column j are

$$a_{ij} + b_{ij} \quad \text{and} \quad ca_{ij},$$

respectively.

Note that for fixed k and n , the set of all k by n matrices satisfies all the properties of a linear space. This fact is hardly surprising, for a k by n matrix is very much like a $k \cdot n$ tuple; that only difference is that the components are written in a rectangular array instead of a linear array.

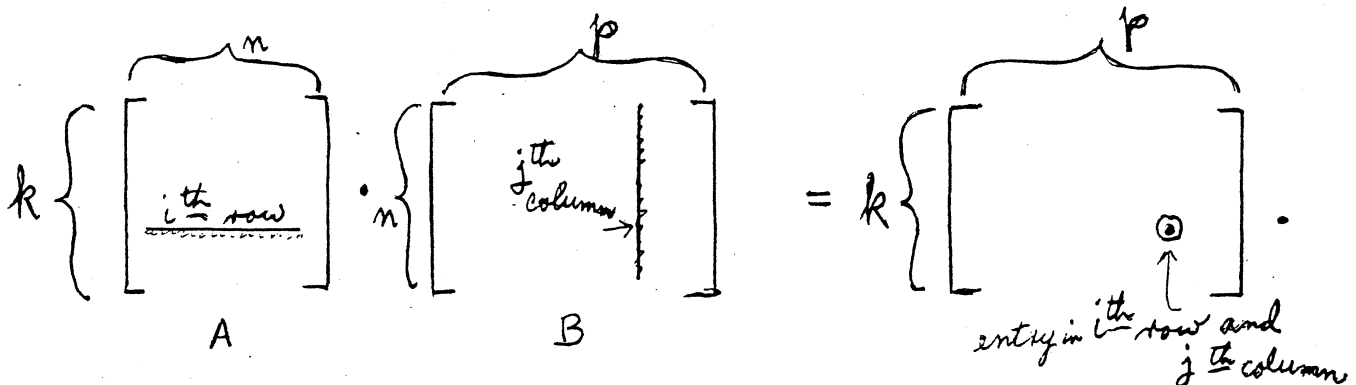
Unlike tuples, however, matrices have a further operation, a product operation. It is defined as follows:

Definition. If A is a k by n matrix, and B is an n by p matrix, we define the product $D = A \cdot B$ of A and B to be the matrix of size k by p whose entry d_{ij} in row i and column j is given by the formula

$$d_{ij} = \sum_{s=1}^n a_{is} b_{sj}.$$

Here $i = 1, \dots, k$ and $j = 1, \dots, p$.

The entry d_{ij} is computed, roughly speaking, by taking the "dot product" of the i^{th} row of A with the j^{th} column of B . Schematically,



This definition seems rather strange, but it is in fact extremely useful. Motivation will come later! One important justification for this definition is the fact that this product operation satisfies some of the familiar "laws of algebra" :

Theorem 1. Matrix multiplication has the following properties: Let A, B, C, D be matrices.

(1) (Distributivity) If $A \cdot (B + C)$ is defined, then

$$A \cdot (B + C) = A \cdot B + A \cdot C .$$

Similarly, if $(B + C) \cdot D$ is defined, then

$$(B + C) \cdot D = B \cdot D + C \cdot D .$$

(2) (Homogeneity) If $A \cdot B$ is defined, then

$$(cA) \cdot B = c(A \cdot B) = A \cdot (cB) .$$

(3) (Associativity) If $A \cdot B$ and $B \cdot C$ are defined, then

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C .$$

(4) (Existence of identities) For each m , there is an m by m matrix I_m such that for matrices A and B , we have

$$I_m \cdot A = A \quad \text{and} \quad B \cdot I_m = B$$

whenever these products are defined.

Proof. We verify the first distributivity formula. In order for $B + C$ to be defined, B and C must have the same size, say n by p . Then in order for $A \cdot (B + C)$ to be defined, A must have n columns. Suppose A has size k by n . Then $A \cdot B$ and $A \cdot C$ are defined and have size k by p ; thus their sum is also defined. The distributivity formula now follows from the equation

$$\sum_{s=1}^n a_{is} (b_{sj} + c_{sj}) = \sum_{s=1}^n a_{is} b_{sj} + \sum_{s=1}^n a_{is} c_{sj}.$$

The other distributivity formula and the homogeneity formula are proved similarly. We leave them as exercises.

Now let us verify associativity.

If A is k by n and B is n by p , then $A \cdot B$ is k by p . The product $(A \cdot B) \cdot C$ is thus defined provided C has size p by q . The product $A \cdot (B \cdot C)$ is defined in precisely the same circumstances. Proof of equality is an exercise in summation symbols: The entry in row i and column j of $(A \cdot B) \cdot C$ is

$$\sum_{t=1}^p \left(\sum_{s=1}^n a_{is} b_{st} \right) c_{tj};$$

and the corresponding entry of $A \cdot (B \cdot C)$ is

$$\sum_{s=1}^n a_{is} \left(\sum_{t=1}^p b_{st} c_{tj} \right).$$

These two expressions are equal.

Finally, we define matrices I_m that act as identity elements.

Given m , let I_m be the m by m matrix whose general entry is δ_{ij} , where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The matrix I_m is a square matrix that has 1's down the "main diagonal" and 0's elsewhere. For instance, I_4 is the matrix

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now the product $I_m \cdot A$ is defined in the case where A has m rows. In this case, the general entry of the product $C = I_m \cdot A$ is given by the equation

$$c_{ij} = \sum_{s=1}^m \delta_{is} a_{sj}.$$

Let i and j be fixed. Then as s ranges from 1 to m , all but one of the terms of this summation vanish. The only one that does not vanish is the one for which $s = i$, and in that case $\delta_{is} = 1$. We conclude that

$$c_{ij} = 0 + \cdots + 0 + \delta_{ii} a_{ij} + 0 + \cdots + 0 = a_{ij}.$$

An entirely similar proof shows that $B \cdot I_m = B$ if B has m columns. \square

Remark. If $A \cdot B$ is defined, then $B \cdot A$ need not be defined. And even if it is defined, the two products need not be equal. For example,

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}, \quad \text{and}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}.$$

Remark. A natural question to ask at this point concerns the existence of multiplicative inverses in the set of matrices. We shall study the answer to this question in a later section.

Exercises

1. Verify the other half of distributivity.
2. Verify homogeneity of matrix multiplication.
3. Show the identity element is unique. [Hint: If I'_m and I''_m are two possible choices for the identity element of size m by m , compute $I'_m \cdot I''_m$.]
4. Find a non-zero 2 by 2 matrix A such that $A \cdot A$ is the zero matrix. Conclude that there is no matrix B such that $B \cdot A = I_2$.
5. Consider the set of m by m matrices; it is closed under addition and multiplication. Which of the field axioms (the algebraic axioms that the real numbers satisfy) hold for this set? (Such an algebraic object is called in modern algebra a "ring with identity.")

Systems of linear equations

Given numbers a_{ij} for $i = 1, \dots, k$ and $j = 1, \dots, n$, and given numbers c_1, \dots, c_k , we wish to study the following, which is called a system of k linear equations in n unknowns:

$$(*) \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2 \\ \dots \\ a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = c_k \end{array} \right.$$

A solution of this system is a vector $X = (x_1, \dots, x_n)$ that satisfies each equation. The solution set of the system consists of all such vectors; it is a subset of V_n .

We wish to determine whether this system has a solution, and if so, what the nature of the general solution is. Note that we are not assuming anything about the relative size of k and n ; they may be equal, or one may be larger than the other.

Matrix notation is convenient for dealing with this system of equations. Let A denote the k by n matrix whose entry in row i and column j is a_{ij} . Let X and C denote the matrices

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} .$$

These are matrices with only one column; accordingly, they are called column matrices. The system of equations (*) can now be written in matrix form as

$$A \cdot X = C.$$

A solution of this matrix equation is now, strictly speaking, a column matrix rather than an n-tuple. However, one has a natural correspondence

$$(x_1, \dots, x_n) \longrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

between n-tuples and column matrices of size n by 1 . It is a one-to-one correspondence, and even the vector space operations correspond. What this means is that we can identify V_n with the space of all n by 1 matrices if we wish; all this amounts to is a change of notation.

Representing elements of V_n as column matrices is so convenient that we will adopt it as a convention throughout this section, whenever we wish.

Example 1. Consider the system

$$2x + y + z = 1$$

$$x - y = 2$$

$$3x + z = 0 .$$

[Here we use x, y, z for the unknowns instead of x_1, x_2, x_3 , for convenience.] This system has no solution, since the sum of the first two equations contradicts the third equation.

Example 2. Consider the system

$$2x + y + z = 1$$

$$x - y = 2$$

$$3x + z = 3$$

This system has a solution; in fact, it has more than one solution. In solving this system, we can ignore the third equation, since it is the sum of the first two. Then we can assign a value to y arbitrarily, say $y = t$, and solve the first two equations for x and z . We obtain the result

$$x = 2 + y = 2 + t$$

$$y = t$$

$$z = 1 - 2x - y = 1 - 2(2+t) - t = -3 - 3t.$$

The solution set consists of all matrices of the form

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+t \\ t \\ -3-3t \end{bmatrix}.$$

Shifting back to tuple notation, we can say that the solution set consists of all vectors X such that

$$X = (x, y, z) = (2+t, t, -3-3t)$$

or

$$X = (2, 0, -3) + t(1, 1, -3).$$

This expression shows that the solution set is a line in V_3 , and in "solving" the system, we have written the equation of this line in parametric form.

Now we tackle the general problem. We shall prove the following result:

Suppose one is given a system of k linear equations in n unknowns. Then the solution set is either (1) empty, or (2) it consists of a single point, or (3) it consists of the points of an m -plane in V_n , for some $m > 0$. In case (1), we say the system is inconsistent, meaning that it has no solution.

In case (2), the solution is unique. And in case (3), the system has infinitely many solutions.

We shall apply Gauss-Jordan elimination to prove these facts. The crucial result we shall need is stated in the following theorem:

Theorem 2. Consider the system of equations $A \cdot X = C$, where A is a k by n matrix and C is a k by 1 matrix. Let B be the matrix obtained by applying an elementary row operation to A , and let C' be the matrix obtained by applying the same elementary row operation to C . Then the solution set of the system $B \cdot X = C'$ is the same as the solution set of the system $A \cdot X = C$.

Proof. Exchanging rows i and j of both matrices has the effect of simply exchanging equations i and j of the system. Replacing row i by itself plus c times row j has the effect of replacing the i^{th} equation by itself plus c times the j^{th} equation. And multiplying row i by a non-zero scalar d has the effect of multiplying both sides of the i^{th} equation by d . Thus each solution of the first system is also a solution of the second system.

Now we recall that the elementary operations are invertible. Thus the system $A \cdot X = C$ can be obtained by applying an elementary operation to both sides of the equation $B \cdot X = C'$. It follows that every solution of the second system is a solution of the first system.

Thus the two solution sets are identical. \square

We consider first the case of a homogeneous system of equations, that is, a system whose matrix equation has the form

$$A \cdot X = \underline{0}.$$

In this case, the system obviously has at least one solution, namely the trivial solution $X = \underline{0}$. Furthermore, we know that the set of solutions is a linear subspace of V_n , that is, an m -plane through the origin for some m . We wish to determine the dimension of this solution space, and to find a basis for it.

Definition. Let A be a matrix of size k by n . Let W be the row space of A ; let r be the dimension of W . Then r equals the number of non-zero rows in the echelon form of A . It follows at once that $r \leq k$. It is also true that $r \leq n$, because W is a subspace of V_n . The number r is called the rank of A (or sometimes the row rank of A).

Theorem 3. Let A be a matrix of size k by n . Let r be the rank of A . Then the solution space of the system of equations $A \cdot X = \underline{0}$ is a subspace of V_n of dimension $n - r$.

Proof. The preceding theorem tells us that we can apply elementary operations to both the matrices A and $\underline{0}$ without changing the solution set. Applying elementary operations to $\underline{0}$ leaves it unchanged, of course.

So let us apply elementary operations to A so as to bring A into reduced echelon form D , and consider the system $D \cdot X = \underline{0}$. The number of non-zero rows of D equals the dimension of the row space of A , which is r . Now for a zero row of D , the corresponding equation is automatically satisfied, no matter what X we choose. Only the first r equations are relevant.

Suppose that the pivots of D appear in columns j_1, \dots, j_r . Let J denote the set of indices $\{j_1, \dots, j_r\}$ and let K consist of the remaining indices from the set $\{1, \dots, n\}$. Each unknown x_j for which j is in J appears with a non-zero coefficient in only one of the equations of the system $D \cdot X = \underline{0}$. Therefore, we can "solve" for each of these unknowns in terms of the remaining unknowns x_k , for k in K . Substituting these expressions for x_{j_1}, \dots, x_{j_r} into the n -tuple $X = (x_1, \dots, x_n)$, we see that the general solution of the system can be written as a vector of which each component is a linear combination of the x_k , for k in K . (Of course, if k is in K , then the linear combination that appears in the k^{th} component consists merely of the single term x_k !)

Let us pause to consider an example.

Example 3. Let A be the 4 by 5 matrix given on p.A20. The equation $A \cdot X = \underline{0}$ represents a system of 4 equations in 5 unknowns. Now A reduces by row operations to the reduced echelon matrix

$$D = \begin{bmatrix} 1 & 0 & -8 & 0 & -3 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the pivots appear in columns 1, 2 and 4; thus J is the set $\{1, 2, 4\}$ and K is the set $\{3, 5\}$. The unknowns x_1 , x_2 , and x_4 each appear in only one equation of the system. We solve for these unknowns in terms of the others as follows:

$$x_1 = 8x_3 + 3x_5$$

$$x_2 = -4x_3 - 2x_5$$

$$x_4 = 0.$$

The general solution can thus be written (using tuple notation for convenience)

$$X = (8x_3 + 3x_5, -4x_3 - 2x_5, x_3, 0, x_5), \text{ or}$$

$$X = (8x_3, -4x_3, x_3, 0, 0) + (3x_5, -2x_5, 0, 0, x_5), \text{ or}$$

$$X = x_3(8, -4, 1, 0, 0) + x_5(3, -2, 0, 0, 1).$$

The solution space is thus spanned by two vectors $(8, -4, 1, 0, 0)$ and $(3, -2, 0, 0, 1)$.

The same procedure we followed in this example can be followed in general. Once we write X as a vector of which each component is a linear combination of the x_k , then we can write it as a sum of vectors each of which involves only one of the unknowns x_k , and then finally as a linear combination, with coefficients x_k , of vectors in V_n . There are of course $n - r$ of the

unknowns x_k , and hence $n - r$ of these vectors.

It follows that the solution space of the system has a spanning set consisting of $n - r$ vectors. We now show that these vectors are independent; then the theorem is proved. To verify independence, it suffices to show that if we take the vector X , which equals a linear combination with coefficients x_k of these vectors, then $X = \underline{0}$ if and only if each x_k (for k in K) equals 0. This is easy. Consider the first expression for X that we wrote down, where each component of X is a linear combination of the unknowns x_k . The k^{th} component of X is simply x_k . It follows that the equation $X = \underline{0}$ implies in particular that for each k in K , we have $x_k = 0$.

For example, in the example we just considered, we see that the equation $X = \underline{0}$ implies that $x_3 = 0$ and $x_5 = 0$, because x_3 is the third component of X and x_5 is the fifth component of X . \square

This proof is especially interesting because it not only gives us the dimension of the solution space of the system, but it also gives us a method for finding a basis for this solution space, in practice. All that is involved is Gauss-Jordan elimination.

Corollary 4. Let A be a k by n matrix. If the rows of A are independent, then the solution space of the system $A \cdot X = \underline{0}$ has dimension $n - k$. \square

Now we consider the case of a general system of linear equations, of the form $A \cdot X = C$. For the moment, we assume that the system has at least one solution, and we determine what the general solution looks like in this case.

Theorem 5. Let A be a k by n matrix. Let r equal the rank of A . If the system $A \cdot X = C$ has a solution, then the solution set is a plane in V_n of dimension $m = n - r$.

Proof. Let $X = P$ be a solution of the system. Then $A \cdot P = C$. If X is a column matrix such that $A \cdot X = C$, then $A \cdot (X - P) = \underline{0}$, and conversely. The solution space of the system $A \cdot X = \underline{0}$ is a subspace of V_n of dimension $m = n - r$; let A_1, \dots, A_m be a basis for it. Then X is a solution of the system $A \cdot X = C$ if and only if $X - P$ is a linear combination of the vectors A_i , that is, if and only if

$$X = P + t_1 A_1 + \dots + t_m A_m$$

for some scalars t_i . Thus the solution set is an m -plane in V_n . \square

Now let us try to determine when the system $A \cdot X = C$ has a solution. One has the following general result:

Theorem 6. Let A be a k by n matrix. Let r equal the rank of A .

(a) If $r < k$, then there exist vectors C in V_k such that the system $A \cdot X = C$ has no solution.

(b) If $r = k$, then the system $A \cdot X = C$ always has a solution.

Proof. We consider the system $A \cdot X = C$ and apply elementary row operations to both A and C until we have brought A into echelon form B . (For the moment, we need not go all the way to reduced echelon form.) Let C' be the column matrix obtained by applying these same row operations to C . Consider the system $B \cdot X = C'$.

Consider first the case $r < k$. In this case, the last row at least of B is zero. The equation corresponding to this row has the form

$$0x_1 + \dots + 0x_n = c'_k,$$

where c'_k is the entry of C' in row k . If c'_k is not zero, there are no values of x_1, \dots, x_n satisfying this equation, so the system has no solution.

Let us choose C' to be a k by 1 matrix whose last entry is non-zero. Then apply the same elementary operations as before, in reverse order, to both B and C' . These operations transform B back to A ; when we apply them to C' , the result is a matrix C such that the system $A \cdot X = C$ has no solution.

Now in the case $r = k$, the echelon matrix B has no zero rows, so the difficulty that occurred in the preceding paragraph does not arise. We shall show that in this case the system has a solution.

More generally, we shall consider the following two cases at the same time: Either (1) B has no zero rows, or (2) whenever the i^{th} row of B is zero, then the corresponding entry c'_i of C' is zero. We show that in either of these cases, the system has a solution.

Let us consider the system $B \cdot X = C'$ and apply further operations to both B and C' , so as to reduce B to reduced echelon form D . Let C'' be the matrix obtained by applying these same operations to C' . Note that the zero rows of B , and the corresponding entries of C' , are not affected by these operations, since reducing B to reduced echelon form requires us to work only with the non-zero rows.

Consider the resulting system of equations $D \cdot X = C''$. We now proceed as in the proof of Theorem 3. Let J be the set of column indices in which the pivots of D appear, and let K be the remaining indices. Since each x_j , for j in J , appears in only one equation of the system, we can solve for each x_j in terms of the numbers c''_i and the unknowns x_k . We can now assign values arbitrarily to the x_k and thus obtain a particular solution of the system. The theorem follows. \square

The procedure just described actually does much more than was necessary to prove the theorem. It tells us how to determine, in a particular case, whether or not there is a solution; and it tells us, when there is one, how to express the solution set in parametric form as an m -plane in V_n .

Consider the following example:

Example 4. Consider once again the reduced echelon matrix of Example 3:

$$D = \begin{bmatrix} 1 & 0 & -8 & 0 & -3 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system

$$D \cdot X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

has no solution because the last equation of the system is

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1.$$

On the other hand, the system

$$D \cdot X = \begin{bmatrix} -1 \\ 3 \\ 7 \\ 0 \end{bmatrix}$$

does have a solution. Following the procedure described in the preceding proof,

we solve for the unknowns x_1 , x_2 , and x_4 as follows:

$$x_1 = -1 + 8x_3 + 3x_5$$

$$x_2 = 3 - 4x_3 - 2x_5$$

$$x_4 = 7.$$

The general solution is thus the 2-plane in V_5 specified by the parametric equation

$$X = (-1, 3, 0, 7, 0) + x_3(8, -4, 1, 0, 0) + x_5(3, -2, 0, 0, 1).$$

Remark. Solving the system $A \cdot X = C$ in practice involves applying elementary operations to A , and applying these same operations to C . A convenient way to perform these calculations is to form a new matrix from A by adjoining C as an additional column. This matrix is often called the augmented matrix of the system. Then one applies the elementary operations to this matrix, thus dealing with both A and C at the same time. This procedure is described in 16.18 of vol. I of Apostol.

Exercises

1. Let A be a k by n matrix. (a) If $k < n$, show that the system $A \cdot X = \underline{0}$ has a solution different from $\underline{0}$. (Is this result familiar?) What can you say about the dimension of the solution space? (b) If $k > n$, show that there are values of C such that the system $A \cdot X = C$ has no solution.

2. Consider the matrix A of p. A23. (a) Find the general solution of the system $A \cdot X = \underline{0}$. (b) Does the system $A \cdot X = C$ have a solution for arbitrary C ?

3. Repeat Exercise 2 for the matrices C , D , and E of p. A23.

4. Let B be the matrix of p. A23. (a) Find the general solution of the system

$$B \cdot X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Find conditions on a, b , and c that are necessary and sufficient for the system $B \cdot X = C$ to have a solution, where $C = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. [Hint: What happens to C when you reduce B to echelon form?]

5. Let A be the matrix of p. A20. Find conditions on a, b, c , and d that are necessary and sufficient for the system $A \cdot X = C$ to have a solution, where

$$C = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

(6.) Let A be a k by n matrix; let r be the rank of A . Let R be the set of all those vectors C of V_k for which the system

$$A \cdot X = C$$

has a solution. (That is, R is the set of all vectors of the form $A \cdot X$, as X ranges over V_n .)

(a) Show that R is a subspace of V_k .

(b) Show that R has dimension r . [Hint: Let W be the solution space of the system $A \cdot X = \underline{0}$. Then W has dimension $m = n - r$. Choose a basis A_1, \dots, A_m for W . By adjoining vectors one at a time, extend this to a basis $A_1, \dots, A_m, B_1, \dots, B_r$ for all of V_n . Show the vectors $A \cdot B_1, \dots, A \cdot B_r$ span R ; this follows from the fact that $A \cdot A_i = \underline{0}$ for all i . Show these vectors are independent.]

(c) Conclude that if $r < k$, there are vectors C in V_k such that the system $A \cdot X = C$ has no solution; while if $r = k$, this system has a solution for all C . (This provides an alternate proof of Theorem 6.)

(7.) Let A be a k by n matrix. The columns of A , when looked at as elements of V_k , span a subspace of V_k that is called the column space of A . The row space and column space of A are very different, but it is a totally unexpected fact that they have the same dimension! Prove this fact as follows: Let R be the subspace of V_k defined in Exercise 6. Show that R is spanned by the vectors $A \cdot E_1, \dots, A \cdot E_n$; conclude that R equals the column space of A .

Cartesian equations of k-planes in V_n .

There are two standard ways of specifying a k-plane M in V_n . One is by an equation in parametric form:

$$X = P + t_1 A_1 + \dots + t_k A_k ,$$

where A_1, \dots, A_k are independent vectors in V_n . (If these vectors were not independent, this equation would still specify an m -plane for some m , but some work would be required to determine m . We normally require the vectors to be independent in the parametric form of the equation of a k -plane.)

Another way to specify a plane in V_n is as the solution set of a system of linear equations

$$A \cdot X = C ,$$

where the rows of A are independent. If A has size k by n , then the plane in question has dimension $n - k$. The equation is called a cartesian form for the equation of a plane. (If the rows of A were not independent, then the solution set would be either empty, or an m -plane for some m , but some work would be required to determine m .)

The process of "solving" the system of equations $A \cdot X = C$ that we described in the preceding section is an algorithm for passing from a cartesian equation for M to a parametric equation for M . One can ask whether there is a process for the reverse, for passing from a parametric equation for M to a cartesian equation. The answer is "yes," as we shall see shortly. The other question one might ask is, "Why should one care?" The answer is that sometimes one form is convenient, and other times the other form is more useful. Particularly is this true in the case of 3-dimensional space V_3 , as we shall see.

Definition. Let A be a matrix of size k by n . Let A_1, \dots, A_k be the rows of A ; let W be the subspace of V_n they span. Now the vector X is a solution of the system $A \cdot X = \underline{0}$ if and only if X is orthogonal to each of the vectors A_i . This statement is equivalent to the statement that X is orthogonal to every vector belonging to W . The solution space of this system is for this reason sometimes called the orthogonal complement of W . It is often denoted W^\perp (read "W perp".)

We have the following result:

Theorem 7. If W is a subspace of V_n of dimension k , then its orthogonal complement has dimension $n - k$. Furthermore, W is the orthogonal complement of W^\perp ; that is, $(W^\perp)^\perp = W$.

Proof. That W^\perp has dimension $n - k$ is an immediate consequence of Theorem 3; for W is the row space of a k by n matrix A with independent rows A_i , whence W^\perp is the solution space of the system $A \cdot X = \underline{0}$.

The space $(W^\perp)^\perp$ has dimension $n - (n - k)$, by what we just proved. And it contains each vector A_i (since $A_i \cdot X = \underline{0}$ for each X in W^\perp .) Therefore it equals the space spanned by A_1, \dots, A_k . \square

Theorem 8. Suppose a k -plane M in V_n is specified by the parametric equation

$$X = P + t_1 A_1 + \dots + t_k A_k,$$

where the vectors A_i are independent. Let W be the space they span; and let B_1, \dots, B_m be a basis for W^\perp . If B is the matrix with rows B_1, \dots, B_m , then the equation $B \cdot (X - P) = \underline{0}$, or

$$B \cdot X = B \cdot P,$$

is a cartesian equation for M .

Proof. The vector X lies in M if and only if $X - P$ belongs to W . This occurs if and only if $X - P$ is orthogonal to each of the vectors B_i , and this occurs if and only if $B \cdot (X - P) = \underline{0}$. \square

The preceding proof actually tells us how to find a cartesian equation for M . One takes the matrix A whose rows are the vectors A_i ; one finds a basis B_1, \dots, B_m for the solution space of the system $A \cdot X = \underline{0}$, using the Gauss-Jordan algorithm; and then one writes down the equation $B \cdot X = B \cdot P$.

We now turn to the special case of V_3 , whose model is the familiar 3-dimensional space in which we live. In this space, we have only lines (1-planes) and planes (2-planes) to deal with. And we can use either the parametric or cartesian form for lines and planes, as we prefer. However, in this situation we tend to prefer:

parametric form for a line, and

cartesian form for a plane.

Let us explain why.

If L is a line given in parametric form $X = P + tA$, then A is uniquely determined up to a scalar factor. (The point P is of course not determined.) The equation itself then exhibits some geometric information about the line; one can for instance tell by inspection whether or not two lines are parallel.

On the other hand, if M is a plane given in parametric form by the equation $X = P + sA + tB$, one does not have as much geometric information immediately at hand. However, let us seek to find a cartesian equation for this plane. We note that the orthogonal complement of $L(A, B)$ is one-dimensional, and is thus spanned by a single non-zero vector

$N = (a_1, a_2, a_3)$. We call N a normal vector to the plane M ; it is uniquely determined up to a scalar factor. (In practice, one finds N by solving the system of equations

$$\begin{aligned} A \cdot N &= \underline{0}, \\ B \cdot N &= \underline{0} .) \end{aligned}$$

Then a cartesian equation for M is the equation

$$N \cdot (X - P) = 0 .$$

If P is the point (p_1, p_2, p_3) of the plane M , this equation has the form

$$(*) \quad a_1(x_1 - p_1) + a_2(x_2 - p_2) + a_3(x_3 - p_3) = 0 .$$

We call this the equation of the plane through $P = (p_1, p_2, p_3)$ with normal vector $N = (a_1, a_2, a_3)$.

We have thus proved the first half of the following theorem:

Theorem 9. If M is a 2-plane in V_3 , then M has a cartesian equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = b ,$$

where $N = (a_1, a_2, a_3)$ is non-zero. Conversely, any such equation is the cartesian equation of a plane in V_3 ; the vector N is a normal vector to the plane.

Proof. To prove the converse, we note that this equation is a system consisting of 1 equation in 3 unknowns, and the matrix $A = [a_1 \ a_2 \ a_3]$ has rank 1. Therefore the solution space of the system $A \cdot X = [b]$ is a plane of dimension $3 - 1 = 2$. \square

Now we see why the cartesian equation of a plane is useful; it contains some geometric information about the plane. For instance, one can tell by inspection whether two planes given by cartesian equations are parallel.

For they are parallel if and only if their normal vectors are parallel, and that can be determined by inspection of the two equations.

Similarly, one can tell readily whether the line $X = P + tA$ is parallel to a plane M ; one just checks whether or not A is orthogonal to the normal vector of M .

Many theorems of 3-dimensional geometry are now easy to prove. us consider some examples.

Theorem 10. Three planes in V_3 intersect in a single point if and only if their normal vectors are independent.

Proof. Take a cartesian equation for each plane; collectively, they form a system $A \cdot X = C$ of three equations in three unknowns. The rows of A are the normal vectors. The solution space of the system (which consists of the points common to all three planes) consists of a single point if and only if the rows of A are independent. \square

Theorem 11. Two non-parallel planes in V_3 intersect in a straight line.

Proof. Let $N_1 \cdot X = b_1$ and $N_2 \cdot X = b_2$ be cartesian equations for the two planes. Their intersection consists of those points X that satisfy both equations. Since N_1 and N_2 are not zero and are not parallel, the matrix having rows N_1 and N_2 has rank 2. Hence the solution of this system of equations is a 1-plane in V_3 . \square

Theorem 12. Let L be a line, and M a plane, in V_3 . If L is parallel to M , then their intersection is either empty or all of L . If L is not parallel to M , then their intersection is a single point.

Proof. Let L have parametric equation $X = P + tA$; let M have cartesian equation $N \cdot X = b$. We wish to determine for what values of t the point $X = P + tA$ lies on the plane M ; that is, to determine the solutions of the equation

$$N \cdot (P + tA) = b.$$

Now if L is parallel to M , then the vector A is perpendicular to the normal vector N to M ; that is, $N \cdot A = 0$. In this case, the equation

$$N \cdot (P + tA) = b$$

holds for all t if it happens that $N \cdot P = b$, and it holds for no t if $N \cdot P \neq b$. Thus the intersection of L and M is either all of L , or it is empty.

On the other hand, if L is not parallel to M , then $N \cdot A \neq 0$. In this case the equation can be solved uniquely for t . Thus the intersection of L and M consists of a single point. \square

Example 5. Consider the plane $M = M(P; A, B)$ in V_3 , where $P = (1, -7, 0)$ and $A = (1, 1, 1)$ and $B = (-1, 2, 0)$. To find a normal vector $N = (a_1, a_2, a_3)$ to M , we solve the system

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ -a_1 + 2a_2 &= 0. \end{aligned}$$

One can use the Gauss-Jordan algorithm, or in this simple case, proceed almost by inspection. One can for instance set $a_2 = 1$. Then the second equation implies that $a_1 = 2$; and then the first equation tells us that $a_3 = -a_1 - a_2 = -3$. The plane thus has cartesian equation

$$2(x_1 - 1) + (x_2 + 7) - 3(x_3 - 0) = 0,$$

or

$$2x_1 + x_2 - 3x_3 = -5.$$

Exercises

1. The solution set of the equation

$$3x_1 + 2x_2 - x_3 = 15$$

is a plane in V_3 ; write the equation of this plane in parametric form.

2. Write parametric equations for the line through $(1,0,0)$ that is perpendicular to the plane $x_1 - x_3 = 5$.

3. Write a parametric equation for the line through $(0,5,-2)$ that is parallel to the planes $2x_2 = x_3$ and $5x_1 + x_2 - 7x_3 = 4$.

4. Show that if P and Q are two points of the plane M , then the line through P and Q is contained in M .

5. Write a parametric equation for the line of intersection of the planes of Exercise 3.

6. Write a cartesian equation for the plane through $P = (-1,0,2)$ and $Q = (3,1,5)$ that is parallel to the line through $R = (1,1,1)$ with direction vector $A = (1,3,4)$.

7. Write cartesian equations for the plane $M(P;A,B)$ in V_4 , where $P = (1, -1, 0, 2)$ and $A = (1, 0, 1, 0)$ and $B = (2, 1, 0, 1)$.

8. Show that every $n - 1$ plane in V_n is the solution set of an equation of the form $a_1x_1 + \dots + a_nx_n = b$, where $(a_1, \dots, a_n) \neq \underline{0}$; and conversely.

9. Let M_1 and M_2 be 2-planes in V_4 ; assume they are not parallel. What can you say about the intersection of M_1 and M_2 ? Give examples to illustrate the possibilities.

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