Solutions to linear systems

1. Consider the system

$$\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 0 \\ 2x & + & y & + & cz & = & 0 \\ 3x & + & y & + & 6z & = & 0. \end{array}$$

a) Take $c = 1$ and find all the solutions.

b) Take $c = 4$ and find all the solutions.

Answer: a) In matrix form we have

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Call the coefficient matrix A. First we check if $\det(A) = 0$.

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 6 \end{vmatrix} = 1(5) - 1(9) + 2(-1) = -6 \neq 0.$$

So, the inverse exists and can be used to find the (unique) solution. We don't actually need to compute the inverse because we know

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) The coefficient matrix is now

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{pmatrix}$$

First we check if $\det(A) = 0$.

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix} = 1(2) - 1(0) + 2(-1) = 0.$$

Since $\det(A) = 0$ there are infinitely many solutions to the *homogeneous* system. We find them by taking a cross product of two rows of A.

$$\langle 1, 1, 2 \rangle \times \langle 2, 1, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = \mathbf{i}(2) - \mathbf{j}(0) + \mathbf{k}(-1) = \langle 2, 0, -1 \rangle.$$

Therefore, all solutions are of the form

$$(x, y, z) = (2a, 0, -a).$$