JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about critical points of functions, how to find them using the first derivatives and how to classify them using the second derivative test. So I have a question here for you about that.

So we have a function w. It's a function of two variables, x and y, and it's given by this polynomial function of them. So w equals x cubed minus 3xy plus y cubed. So what I'd like you to do is to first find the critical values of this function and then classify them-- are they minima or maxima or saddle points-- using the second derivative test. So why don't you pause the video, take some time to work that out. Come back and we can work it out together.

Hopefully, you had some luck working out the solution to this question. Let's have a go at it. So in order to find the critical points, we need to look at the first derivative. So the critical points are the points where both partial derivatives-- or all partial derivatives, if we had a function of more variables-- are equal to zero. So we need to look at the first partials. So the first partials here, w sub x, the partial with respect to x-- well, it's just a polynomial so it's easy to compute those partial derivatives. It's going to be 3 x squared minus 3y, and then the last term gets killed because we treat y as a constant, and so we want that to be equal to zero. And similarly, we want the first partial with respect to y, w sub y, to be equal to zero. And so that's minus 3x plus 3 y squared equals 0.

Now, luckily, these are fairly simple equations, so to solve them, we could, for example, take the first equation and we could solve the first equation for y in terms of x. So that'll give us y equals x squared. And now if we plug y equals x squared into this second equation, well, we get minus 3x plus 3 x squared squared-- so that's x to the fourth-- is equal to 0, and we can divide out by that 3, so that means minus x plus x to the fourth equals 0. Well, OK, so we could have x equal to 0, or you can divide, and then you get x cubed equals 1, and that has solution x equals 1. So x equals 0 or 1. Those are the only solutions to this equation. And then the corresponding y-values, well, we know y is equal to x squared, so this gives us critical points when x is 0, y is 0, and when x is 1, y is 1. So this function has two critical points: (0, 0) and (1, 1).

Now we need to figure out whether those critical points are minima, maxima, saddle points, sum of several of those. So in order to do that, we're going to use this nice tool that we have: the second derivative test. So in order to apply the second derivative test, the first thing I need is the second derivatives. So let's compute them.

So the second-- let's do the xx first. So we take our first partial, 3 x squared minus 3y, and we take another partial of it with respect to x. So that's, in this case, that's just going to be 6x. And then we've got the other pure second partial, yy, so we go back over here, and we look at what our first partial w_y was, and then we take another partial of this with respect to y, so that's just going to be 6y. And then we have the mixed partials w_xy and w_yx, which, of course, are equal to each other whenever our function is nicely behaved, like a polynomial. So w_xy, we just take the two mixed partials and-- OK, so we take the partial of w_x with respect to y, for example, and that gives us minus 3. So these are our three partials, and then often, we, you know, call this one A and this one C and this one B. I guess I kind of mixed up the order a little bit there.

So we look at these three expressions, and now we want to look at what sometimes people call the discriminant, although I don't know if Professor Auroux used that term. So we want to study what the expression A*C minus B squared is, so we want to know is this positive, is this negative? At the critical points. So at the critical points, right? This is important. At the critical points.

So let's do the point (1, 1) first. So at (1, 1), we have A is equal to-- well, we put in x is 1, y is 1 into the expression for A here, and that just gives a 6. We put x 1, y 1 into the expression for C, and that also gives a 6, we put x 1, y 1 into the expression for B, and that gives us minus 3. So A is 6, B is minus 3, C is 6, so A*C minus B squared is equal to, well it's equal to 36 minus 9, so that's 27. And, in particular, it's positive.

So when this is positive, that means we either have a maximum or a minimum. So in order to figure out whether we have a maximum or a minimum, we check the sign of A. So in this case, the sign of A is positive. A is a positive number. So when you have that A*C minus B squared is positive and A is positive, that means you have a minimum. So the critical point (1, 1) is a local minimum for this function.

All right. Now we can do the same thing for the critical point (0, 0). So recall A was 6x, so at (0, 0). A is equal to 0, B was equal to negative 3 everywhere, and C was equal to 6y, so at (0, 0), that's 6 times 0 so that's also 0. So then our quantity that we want to look at, A*C minus B squared, well, that's 0 times 0 minus 9, so that's equal to negative 9. And negative 9 is less than 0, so when A*C minus B squared is less than 0, that means we have a saddle point.

So in this case, the second derivative test was able successfully to distinguish what kinds of critical points we had, and it found that the first critical point 1, 1 was a minimum and that the second critical point 0, 0 was a saddle point.

So just to quickly rehash what we did, we had a function. Back over here, we started with this function w. We had a nice formula for it. We computed its first derivatives. We set them both equal to zero and we solved that system of equations. So we found two solutions to that system of equations, and those two solutions are the critical points, the points where both partial derivatives are equal to zero.

So when you have the two critical points, then you want to apply the second derivative test to figure out for each critical point whether it's a saddle point, a minimum or a maximum. So we took our two critical points, (1, 1) and (0, 0), and at those points, we evaluated the second derivative. So A is the xx second derivative. B is the mixed partial w_xy, and C is the yy second derivative. So we evaluate those expressions at the points in question, and then we look at A*C minus B squared. And then the sign of A*C minus B squared, if it's negative, that gives us a saddle point. If it's positive, that gives us either a maximum or a minimum, and we check which one by looking at the sign of A. So here A was positive, so we got a minimum at (1, 1). So I'll stop there.