Definition of double integration

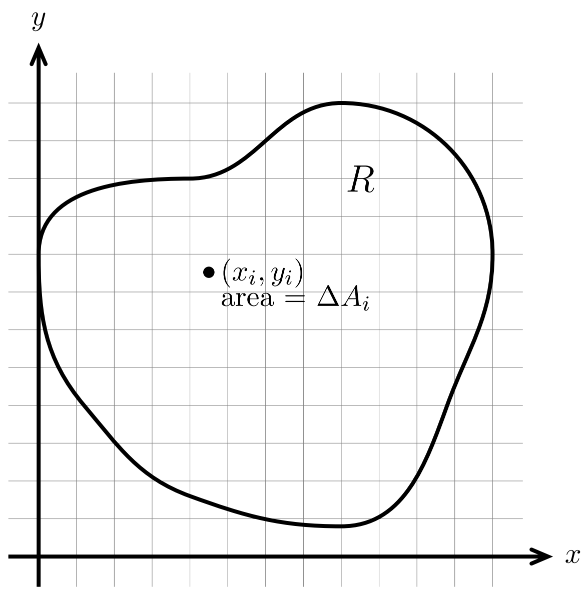
In this note we will work abstractly, defining double integration as a sum, technically a limit of Riemann sums. It is best to learn this first before getting into the details of computing the value of a double integral –we will learn how to do that next.

Definition of double integrals
Suppose we have a region in the plane R and a function $f(x, y)$, Then the double integral

$$\int\int_R f(x, y)\, dA$$

is defined as follows.

Divide the region R into small pieces, numbered from 1 to n. Let ΔA_i be the area of the i^{th} piece and also pick a point (x_i, y_i) in that piece. The figure shows a region R divided into small pieces and shows the i^{th} piece with its area, and choice of a point in the little region.



Now form the sum

$$\sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i,$$

and then, finally

$$\int\int_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i.$$

Here, the limit is taken by letting the number of pieces go to infinity and the area of each piece go to 0. There are technical requirements that the limit exist and be independent of the specific limiting process. In 18.02 these requirements are always met. (Later you might study fractals and other strange objects which don't satisfy them.)

Interpretations of the double integral

As you saw in single variable calculus, these sums can be used to compute areas, volumes, mass, work, moment of inertia and many other quantities. Again, before focusung on some computational issues we will show you how easy it is to setup a double integral to compute certain quantities.

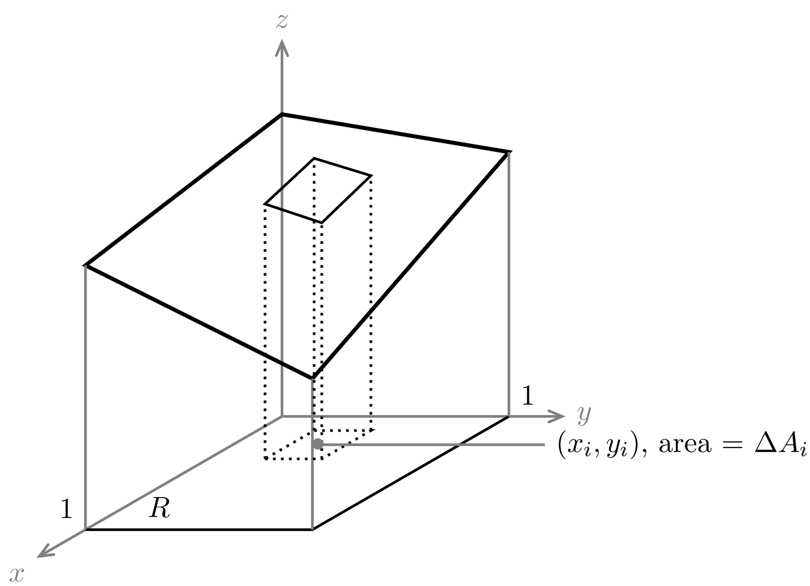
Example 1: Set up a double integral to compute the area of a region R in the plane.

Answer: Use the figure above for visualization. The area of R is just the sum of the areas of the pieces. That is,

$$\text{area} \;=\int\int_R dA.$$

Example 2: Set up a double integral to compute the volume of the solid below the graph of $z = f(x,y) = 2 - .5(x+y)$ and above the unit square in the xy-plane.

Answer: The figure below shows the graph of $f(x,y)$ above the unit square in the plane. The unit square is labeled R. We also show a little piece of the R and the solid region above that piece. We are imagining we've divided R into n small pieces and this is the i^{th} one. It contains the point (x_i, y_i) and has area ΔA_i.



The small solid region is almost a box and so its volume, ΔV_i, is roughly its base times its height, i.e.,

$$\Delta V_i \approx \Delta A_i \times f(x_i, y_i).$$

The total volume is the sum of the volumes of all the small pieces, i.e.,

$$\text{volume} \;= \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} \Delta A_i \times f(x_i, y_i).$$

In the limit this becomes an exact integral for volume

$$\text{volume} \;=\int\int_R f(x,y)\, dA.$$