CHRISTINE BREINER: Welcome back to recitation. In this video I'd like us to work on the following problem. What values of b will make this vector field F a gradient field, where F is determined by e to the x plus y times x plus b i plus x*j? So the e to the x plus y is in both the i component and the j component. And then once you've determined what values b will make that a gradient field, for this b-- or I should've said these b's-- find a potential function f using both methods from the lecture. So why don't you pause the video, work on this, and then when you are ready to look at how I do it bring the video back up.

OK. Welcome back. So I'm going to start off working on the first part of this problem, which is to find the values of b that will make this vector field F a gradient field. And to clarify things for myself, I'm going to write down what M and what N are based on F. So just to have it clear, M is equal to e to the x plus y times x plus b and N is equal to x times e to the x plus y. So those are our values for M and N.

And now if I want f to be a gradient field, what I have to do is I have to have M sub y equal N sub x. So I'm going to determine M sub y and I'm going to determine N sub x and I'm going to compare them and see what value of b I get. So M sub y, fairly straightforward because this is a constant in y. And the derivative of this in terms of y is just this back. Right? It's an exponential function with the value that it has in y is linear. So you get exactly that thing back. So it actually is just e to the x plus y times x plus b. So the derivative of M sub y is just itself. The derivative of M with respect to y, sorry. Not the derivative of M sub y. OK. That's an x. Let me just rewrite that.

OK, now N sub x is going to have two parts. N sub x, the derivative with respect to x of this is 1. And so I have an e to the x plus y. And the derivative with respect to x of e to the x plus y is just e to the plus y, for the same reason as the derivative with respect to y was the same. So then I'm just going to get a plus x e to the x plus y. So that means if I factor that out, I get an e to the x plus y times 1 plus x. And we see that if F is going to be a gradient field then b has to equal 1. Because it can only have one value, and so b has to equal 1. To get N sub x to equal M sub y, b has to equal 1. So now what I'm going to do is I'm going to erase that b, put in a 1, so that the rest of my calculations refer to that.

So now the second part said for this b find a potential function f using both methods from the lecture. So we're going to go through both methods and hopefully we get the same answer both times. So let me come back here. The first method is where I'm integrating along a curve from (0, 0) to (x_1, y_1). So I'm going to do it in the following way. I'm going to let C_1-- so here's (0, 0)-- I'm going to let C_1 be the curve from (0, 0) up to (0, y_1). And then C_2 be the curve-- so it's parameterized in that direction-- C_2 be the curve from (0, y_1) to (x_1, y_1). OK? So that's what I'm going to do, and I'm going to let C equal the curve C_1 plus C_2. So I'm going to have C be the full curve. And what I'm interested in doing is finding f of x_1, y_1, which will just equal the integral along C of F dot dr.

So now we need to figure out some important things about C_1 and C_2. What's happening on C_1 and what's happening on C_2. And the first thing I want to point out-- actually, before I do that, let me remind you that this is going to be the integral on C of M*dx plus N*dy. So this will be helpful to refer back to. That's really what we're also doing here. So on C_1, what do I notice? On C_1, x is 0 and dx is 0. And y goes between 0 and y_1. And then on C_2, y is equal to y_1. So dy is equal to 0 and x is going between 0 and x_1. So those are the values that are important to me.

So if you notice from this fact and this fact, we see that if we look at the integral just along C_1, there's going to be no M*dx term. And if we look at the integral along C_2, there's going to be no dy term because of that. So let me write down the terms that do exist, and we'll see some other things drop out also. If I look along first just C_1, I'm only going to get-- I said the dy term, which-- let me just make sure-- dx is 0. I'm only going to get the dy term, which is-- well, x is 0 there. So I'm going to get 0 times e to the 0 plus y, dy. From 0 to y_1. Well that's nice and easy to calculate, thank goodness. That's just 0.

So all I have to do for this one is just leave it at 0. That's everything that happens along C_1. That's what I'm interested in. I just get 0 there. And if I integrate along C_2, as I mentioned, dy is 0. So we don't have any component with N*dy. We just have the component M*dx that we're integrating. OK, so if I integrate along C_2, I just have M*dx and M is e to the x plus y times x plus 1. And y is fixed at y_1. So it's e to the x plus y1 times x plus 1 dx. And I'm going from 0 to x_1. I'm going to make sure I didn't make any mistakes. I'm going to check my work here. Yes, I'm looking good.

OK. So this one is 0. So all I have to do is find an antiderivative of this. And the term-- if I multiply through, I see that here I get exactly the same thing when I look for an antiderivative. And here I get, I believe, two terms when I look for an antiderivative. But I'm going to get some cancellation. And ultimately, when I'm all done I'm going to get this. x e to the x plus y_1 evaluated at 0 and x_1. You could do this. This is really now a single variable problem. So I'm not going to work out all the details, but you might want to do an integration by parts on that first part of it, if that helps. Or an integration by parts on the whole thing. That would also do the trick.

So what do I get here? Then I get x_1 e to the x_1 plus y_1. And then when I put in 0 for x here, I get 0, so that's it. So this, plus possibly a constant, is equal to my f. So I see that in general I get f of x, y is equal to x e to the x plus y plus a constant. So that's what I get in the first method. So now let's use the second method. So I should say f of x_1, y_1.

In the second method, what I do is-- M, remember, is equal to f sub x. So f sub x is equal to M which is equal to e to the x plus y times x plus 1. So if I want to find an antiderivative-- if I want to find f, I should take an antiderivative, right? With respect to x. And so notice I already did that, actually. If I just put this as y, I already did that here. And so I should get something that looks like this: x e to the x plus y plus possibly a function that only depends on y.

And the reason is when I take a derivative with respect to x of this, obviously this would be 0. So it doesn't show up over here. So this, we make sure that-- oh, that, I shouldn't write equals. Sorry. That, I shouldn't write equals. OK? This would imply that this is equal to f. Sorry about that. f sub x was equal to M was equal to this. That implies-- when I take an antiderivative of an x-- that x e to the x plus y plus g of y is equal to f. So I apologize. That wouldn't have been an equals because obviously those two things are not equal. That would imply, I think-- yeah, that would imply something very bad mathematically. So make sure you understand I put an equals sign where I should not have. This is actually a derivative of that. So this is antiderivative of this.

So now I have a candidate for f. And so now I'm going to take the derivative of that. And what's the derivative of that with respect to y? So f sub y based on this is going to be equal to x e to the x plus y plus g prime of y. So the prime here indicates it's in a derivative in y. And now that f sub y should also equal N. And N equals x e to the x plus y.

So what do I get here? I see x e to the x plus y has to equal x e to the x plus y plus g prime of y. Which means g prime of y is equal to 0. Which means when I take an antiderivative of that I just get a constant. That means g of y was a constant. So that implies that this boxed expression right here is f of x, y if g of y is just a constant.

So let me go through that logic one more time. I had f sub x. I took an antiderivative to get f but it involved a constant in x that was a function of y. I take a derivative of that in y. I compare that to what I know the derivative is in y. That gives me that this is 0. So its antiderivative, which is g of y, is just a constant. And so altogether this implies that f of x, y is equal to x e to the x plus y plus a constant. Which is exactly what I got before. Fortunately, I got two answers that are the same. So that's it. I'll stop there.