Finding Area Using Line Integrals

Use a line integral (and Green's Theorem) to find the area of the unit circle.

Answer: Recall that Green's Theorem tells us $\oint_C M \, dx + N \, dy = \iint_R N_x - M_y \, dA$. To find the area of the unit circle we let $M = 0$ and $N = x$ to get $\iint_R 1 \, dA = \oint_C x \, dy$.

We parametrize the circle by $x = \cos\theta$, $y = \sin\theta$, $0 < \theta \leq 2\pi$, so $x \, dy = \cos^2\theta \, d\theta$. Then

$$
\begin{aligned}
\text{Area} \quad &= \quad \iint_R 1 \, dA \\
&= \quad \oint_C x \, dy \\
&= \quad \int_0^{2\pi} \cos^2\theta \, d\theta \\
&= \quad \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta \\
&= \quad \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi} \\
&= \quad \pi.
\end{aligned}
$$

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