Problems: Extended Stokes' Theorem

Let $\mathbf{F} = \langle 2xz + y, 2yz + 3x, x^2 + y^2 + 5 \rangle$. Use Stokes' theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve shown on the surface of the circular cylinder of radius 1.

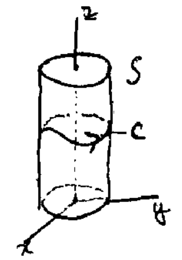

Figure 1: Positively oriented curve around a cylinder.

Answer: This is very similar to an earlier example; we can use Stokes' theorem to calculate this integral even though we don't have an exact description of C. We just make C into part of the boundary of a surface, as shown in the figure below.

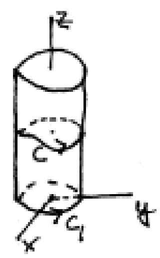

Figure 2: Curves C and C_1 bound part of a cylinder.

Let C_1 be the unit circle in the xy-plane oriented to match C and S the portion of the cylinder between C and C_1. Then Stokes' theorem says:

$$\oint_{C_1 - C} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS.$$

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz + y & 2yz + 3x & x^2 + y^2 + 5 \end{vmatrix} = 2\mathbf{k}.$$

Since the normal vector to S is always orthogonal to $\mathbf{k}$, $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} = 0$.
Thus, $\oint_{C_1 - C} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

To finish, parametrize C_1 by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leq t < 2\pi$ and calculate:

$$\begin{aligned}
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \oint_C (2xz + y)dx + (2yz + 3x)dy + (x^2 + y^2)dz \\
&= \int_0^{2\pi} \sin t(-\sin t\, dt) + 3\cos t(\cos t\, dt) \\
&= \int_0^{2\pi} -1 + 4\cos^2 t\, dt \\
&= \left[-t + \frac{4}{2}(t + \sin t \cos t)\right]_0^{2\pi} = 2\pi.
\end{aligned}$$