CHRISTINE BREINER: Welcome back to recitation. In this video, what I'd like us to do is work on understanding simply connected regions in three dimensions. Well, there's one two-dimensional one, but the rest are three dimensions. So what I want you to do is for each of the following-- there are six different regions-- determine whether or not each of them is simply connected.

So the first one is R^3. The second one is if I take R^3 and I remove the entire z-axis. The third one is if I take R3 and I remove 0. The fourth one is if I take R^3 and remove a circle. The fifth one is R^2 minus a line segment. And the sixth one is a solid torus.

So a solid torus looks like a doughnut, and it includes the inside of the doughnut. This looks like a doughnut, hopefully, to you. And it's not hollow. It includes the inside.

So what I'd like you to do, again, is determine whether or not each of these regions is simply connected. And why don't you pause the video while you work on that. And then bring the video back up when you're ready to check your work.

OK, welcome back. So again, what we're interested in doing is understanding simply connectedness in another dimension. We did something already, a while back, with two dimensions, and so now we want to understand it better in three dimensions. So let's work through these.

Well, I'm not going to write anything down for number one, because you should already know that R^3 is simply connected. But if you weren't sure about it, you could think, any closed curve I draw in R^3, I can certainly get all of the inside of it contained in R^3. Another way to think about it is that I can take that curve and I can collapse it down to a point, and remain in R^3. So then the first one is an easy yes to simply connectedness. OK?

So let's start on the second one, and I'm going to draw a little picture for us. So the second one is R^3. I should go this way. This is x, y, and z, but then I remove the entire z-axis. So I should make this really dark so we know we're removing that part from R3. And I'm removing it all the way up to minus infinity in the z-direction and plus infinity in the z-direction.

Now, the question is can I find any closed curve, that when I try and compress that closed curve down to a point, I can't do it while remaining inside this region that is all of R3 minus the z-axis. And the answer is there is a whole family of curves that do this. If I take a curve that goes around the z-axis, you'll notice that there's something on the inside of it, regardless of--

you know, if I slide it up or down, there's a point on the inside of this curve that is not in the region I'm interested in. The region, again, is R^3 minus the z-axis.

So there are two ways to think about this. You can think about, if I were to take this curve and I were to put a surface across this curve, so it was like a disk, there would be a point on the z-axis that would intersect it. Or you can think about it as saying, I have this curve and if I try and squeeze it down to as small as I can get it, I can't get it as small is I want without hitting the z-axis at some point. The z-axis is kind of in the way, right?

Now, number three is a little different situation. Because in number three, I think this exact same picture, but instead of removing the whole z-axis, I just remove the origin. So let me try and draw a picture of that. So I'm going to make this-- there's a big open circle at the origin. That's not included in our domain, in our region.

So our region is all of R^3 except the origin. And in two-dimensional space, this was not simply connected. But in three-dimensional space it is simply connected. So this is a little different situation than what you had previously.

And so the idea is here, if I take a curve, even if I take a curve that's sitting in the xy-plane that goes around the origin, the point is I can keep this curve in three-dimensional space, and I can wiggle it around, so that I can shrink it down to a point, and the origin doesn't get in the way. It doesn't keep me from doing that. So actually, this region, even though in two-dimensional space it was not simply connected, in three-dimensional space it is. And let's see if we understand the difference.

The difference is in two-dimensional space, if I drew a curve on the xy-plane around the origin, and I wanted to squish it down to a point, the only way to do that would be to bring the curve somehow through the origin. Right? I would be stuck having to pass the curve through the origin to shrink it down to a point. But in three-space, I have another dimension.

So a curve that sits on the xy-plane, I can just kind of lift it a little bit away from the origin, and then I can shrink it down to a point without the origin getting in the way. So having that extra dimension means even though I remove one point, it's still actually a simply connected region. So maybe this is the first place we see that in the three dimensions we have a different case than we had in two dimensions, removing the same kind of object.

So I realize now I haven't been writing down whether these are simply connected or not. So I should write down this is simply connected. And maybe for number two I should go back and formally write not simply connected. So that we have this for posterity.

Now the fourth one is R^3 minus a circle. So let me see if I can draw a picture of that. And the circle, it doesn't really matter where it is. I'm just going to draw one somewhere. So here's my circle. So everything is in my region except this circle.

And the question: is it simply connected? And the answer is: no, the region is not simply connected, because of one particular problem. It's actually the same kind of problem you have when you remove the z-axis.

And that is, if I draw a curve that goes around this circle-- any curve that goes around this circle-- notice that any way I try and move this curve and shrink it down to a point, this circle is going to get in the way for the same reason that the z-axis got in the way. Because this circle is closed, I can't slide the curve I'm interested in away from the circle and then shrink it down. OK. There's some sort of obstruction right here.

And so it's fundamentally different than the case where we just had the origin, because we could take any curve and we could move it away from the origin, and then shrink it down to a point. And the origin didn't get in the way. But here, anywhere I try and move this curve, it's going to have to hit the circle if I want to move it away so I can shrink it to a point in my region. So this circle is preventing me from shrinking it down.

OK, and then there are two more. And the fifth one is R^2 minus a line segment. So now we're in two-dimensional space. OK, and let me just pick a segment. OK. Now, this one is interesting.

Oops. Again I did it. I forgot to write whether it's simply connected or not. Let me come back over to four for posterity. Not simply connected. OK, sorry about that.

The fifth one, because I'm in two dimensions, it's going to be not simply connected, but if I add a third dimension, it would become simply connected. So I want to explain why it's not simply connected here, and then I want to show you why in a third dimension it becomes simply connected. OK?

The problem curves are the curves that do this, that go around this line segment. Because notice, if I want to try and contract this curve down to a point and I don't want to intersect that line segment, in order to do it I'd actually have to move it away from the line segment. I'd have to pass through the line segment. At some point, this curve would intersect that segment in order to be able to shrink it to a point in the region I'm interested in.

So this segment is getting in the way-- we can think of it that way-- of allowing me to contract this down to a point. Actually also, when we talked about simply connectedness in two dimensions, it was easier. Because we could say, if we take any curve and we look at the disk that's spanned by this curve-- where the boundary is this curve, and we look at the region the curve encloses-- notice that this segment is in that region. And there's no way of drawing this kind of curve without the segment being in that region, and that's how we know it's not simply connected.

Now, in three dimensions, what happens? What if I took this exact same picture and I just made the z-axis come out from the board? Why is that suddenly simply connected, whereas in the two-dimensional case it's not?

And the reason is because in this same picture, I could take this same curve, and I could take this shaded thing, and I could push the shaded thing out of the xy-plane. And so I'd still have the same boundary curve, but I'd have the shaded portion not hitting the segment. And so I can find some surface with this boundary that doesn't have this segment in the interior of the surface. And that's another way of thinking about simply connectedness.

So in the two-dimensional case, it is not simply connected, but if I were to add a third dimension, this region would become simply connected. OK. Because I would have no problem for any curve finding some surface that had that curve as a boundary that didn't intersect that segment. So I could keep the surface in the region I was interested in. OK. So that would tell me it was simply connected.

And then the last one is a solid torus. OK, and this one, we might not have dealt with solid tori before, but this is an interesting problem. OK, so there are fundamentally-- we say in math-- that there are two classes of curves that are interesting. We won't get into the exact terminology of what's happening, but there are two types of curves on the torus.

One type of curve is the kind that goes around right here. OK. So it loops around the doughnut in that direction. But that type of curve is nice, because notice, that if I look at the surface in there, it's all inside the solid torus. So that's good. So that seems like that's a curve that promotes simply connectedness. Or it's not telling us it's not simply connected. We'll say that.

But there's another class of curves in the torus. And that's the class of curves that goes around-- this is a little harder to draw, but say around the top, but around the hole. OK? Around the hole. Now any surface I have that I try to draw-- any surface that's going to have that curve as a boundary-- is at some point forced to leave the solid torus. And the reason is really because of the hole in the middle. Right? That's really the reason it happens. OK.

And so you can see the part right in here is on the surface, but it's not in the solid torus. So because I have a curve that any surface I draw that has that curve as a boundary is forced to leave the solid torus, it's a non-simply-connected region. So we say not simply connected.

OK. So I'm going to go back through real quickly and just remind us what was happening. And maybe use the language I was using at the end to describe the first examples, because that might help a little better. So let's go back to the first examples.

OK, in the R^3 example, again, number one, we know it's simply connected. We're not going to worry about it. OK.

But let me draw-- in number two, maybe if I draw some shaded region, this will help us understand it a little bit better. Number two we established was not simply connected. And if you think about it, if you have a curve that goes around the z-axis, and you want to look at a surface that has that curve as its boundary, this surface certainly intersects the z-axis.

The question is, can I keep this curve the way it is, and pull the surface away and have it not intersect the z-axis? And the answer is no. Any way I move the inside of the curve-- basically, what looks like a disk-- it's still going to intersect the z-axis somewhere. Right? And so it's definitely not simply connected.

And the thing I was trying to point out in number three, that it is simply connected, is if I shade the boundary of a curve sitting in the xy-plane, and then I take that shaded disk and I push it up a little, then it no longer hits the origin. And I haven't fundamentally changed my curve at all. And so that's a way of understanding that it is actually simply connected. OK?

So there are a couple of ways to think about it. And without being incredibly mathematically precise, these are some of the best ways we have of thinking about understanding simply connected or not simply connected.

So again, we had six examples. Removing the z-axis from R^3 was not simply connected.

Removing the origin from R^3 was still simply connected. Removing a circle from R^3 was not simply connected for the same reason as the z-axis problem, because here was my disk, and any way I try to move this shaded surface, I can't keep it from intersecting this circle. And then number five was R^2 minus a segment. It was not simply connected, but if I add another dimension, it is simply connected, for the same kind of reason that R^3 minus the origin was. And then number six was the solid torus. Which now, it's kind of hard to see what the solid torus looks like. But we said, there's one kind of curve that behaves fine, but the curve that goes all the way around the hole shows it's, in fact, not simply connected.

So hopefully that was informative, and that's where I'll stop.