Section II Solutions

2A-1 a) This is true because D^2, pD, and multiplich by q are all linear operators:

$$q(y_1+y_2) = qy_1 + qy_2 \quad (1)$$

$$pD(y_1+y_2) = p(Dy_1 + Dy_2)$$
$$\therefore \quad = pDy_1 + pDy_2 \quad (2)$$

$$D^2(y_1+y_2) = D^2y_1 + D^2y_1 \quad (3)$$

Adding ①, ②, ③ gives

$$L(y_1+y_2) = Ly_1 + Ly_2$$

The proof for $L(cy_1) = cLy_1$ is similar.

b) (i) $Ly_h = 0$ since y_h solves the eqn $Ly=0$

$Ly_p = r$ since y_p solves the original eqn.

Adding, using linearity of L: $L(y_h+y_p) = r$ $\quad \therefore y_h + y_p$ is a sol'n.

(ii) if y_1 is any soln, then

$$L(y_1 - y_p) = Ly_1 - Ly_p = r - r = 0$$

$$\therefore \quad y_1 - y_p = y_h \quad (\text{a sol'n of } Ly=0)$$

$$\therefore \quad y_1 = y_h + y_p.$$

Parts (i) + (ii) together show all solns are of the form $y_h + y_p$.

2A-2 a)

$$y = c_1e^x + c_2e^{2x}$$
$$y' = c_1e^x + 2c_2e^{2x}$$
$$y'' = c_1e^x + 4c_2e^{2x}$$

$$y' - y = c_2e^{2x}$$
$$y'' - y' = 2c_2e^{2x}$$
$$\therefore \quad y'' - y' = 2(y' - y)$$

or: $\boxed{y'' - 3y' + 2y = 0}$.

b) The question is whether we can find values for c_1, c_2 such that

$$c_1e^{x_0} + c_2e^{2x_0} = y_0$$
$$c_1e^{x_0} + 2c_2e^{2x_0} = y_0'$$

These equations can be solved (by Cramer's rule) for c_1, c_2 provided that $\begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} \neq 0$. (coefficient determinant)

But this det $= e^{3x_0} \neq 0$ for any x_0.

2A-3 a)

$$y = c_1x + c_2x^2$$
$$y' = c_1 + 2c_2x$$
$$y'' = 2c_2$$

You want to eliminate c_1, c_2. One way —:

$$\begin{cases} c_2 = y''/2 & \text{from last eqn} \\ c_1 = y' - y''x & \text{from 2nd + 3rd eqn.} \end{cases}$$

Substitute into 1st eqn, get

$$y = (y' - y''x)x + \frac{y''}{2}x^2,$$

which by algebra becomes

$$\boxed{x^2y'' - 2xy' + 2y = 0}$$

b) all solns $y = c_1x + c_2x^2$ satisfy $y(0) = 0$

c) This theorem requires that when eqn is written $y'' + p(x)y' + q(x)y = 0$, that p, q be continuous functions. But here, the ODE in standard form is

$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0;$$

coefficients are discontinuous at $x=0$.

2A-4 a) Suppose y_1 is a solution to

$$y'' + p(x)y' + q(x)y = 0 \quad \circledast$$

tangent to x-axis at the pt. x_0. Then

$$y_1(x_0) = 0$$
$$y_1'(x_0) = 0.$$

But $y_2(x) \equiv 0$ is another soln to $\circledast$ with this same property:

$$y_2(x_0) = 0$$
$$y_2'(x_0) = 0$$

$\therefore$ by the uniqueness theorem,

$$y_1 \equiv y_2 \quad \text{for all } x,$$

i.e., $y_1 \equiv 0$.

b)

$$y = x^2$$
$$y' = 2x$$
$$y'' = 2$$

$\therefore \boxed{xy'' - y' = 0}$ is such an equation

or: $\boxed{y'' - \frac{1}{x}y' = 0}$

Part (a) is not contradicted, since the coefficient $\frac{1}{x}$ is discontinuous at $x=0$.

$\boxed{2A-5}$ a) $W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$

$= (m_2 - m_1)e^{(m_1+m_2)x}$;

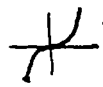
since $e^x \neq 0$ for all x, this is never 0 if $m_1 \neq m_2$. $\therefore$ functions are lin. ind.

b) $W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & mxe^{mx} + e^{mx} \end{vmatrix}$

$= e^{2mx} \neq 0$ for any x.

(This holds true even if $m = 0$).

$\therefore$ the functions are lin. indep't.

$\boxed{2A-6}$ (The graph of $x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$: )

a) If $x \geq 0$, $W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} \equiv 0$

if $x \leq 0$, $W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0$

b) Suppose they were linearly dependent on an interval (a, b) containing 0, that is, suppose there are c_1, c_2 such that $\boxed{c_1y_1 + c_2y_2 = 0 \text{ for all } x \in (a,b)}$.

Then if $x \geq 0$, $y_1 = y_2$, $\therefore c_1 = -c_2$

if $x < 0$, $y_1 = -y_2$, $\therefore c_1 = c_2$

Thus $c_1 = 0$ and $c_2 = 0$, so that y_1 and y_2 are not lin. dep't on (a, b).

Since $y_2' = 2x$ for $x > 0$,

$y_2' = -2x$ for $x < 0$

graph of y_2' is 

Thus y_2'' does not exist at $x = 0$, so it cannot be the solution to a 2^{nd} order equation $y'' + p(x)y' + q(x)y = 0$ on the interval (a, b) containing 0.

Thus thm in the book ($W \equiv 0$ for 2 solns to ODE $\Rightarrow$ solns are lin dept) is not contradicted.

$\boxed{2A-7}$ a) This can be done directly, by differentiating $y_1y_2' - y_1'y_2$. (*see below)

An elegant way to do it is to use the formula for differentiating a determinant: diff. one row at a time, then add:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}' = \begin{vmatrix} u_1' & u_2' \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1' & v_2' \end{vmatrix}$$

(this works for dets. of any size).

Applying this to the Wronskian:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix};$$

since y_1 and y_2 solve $y'' = -py' - qy$, we get the above right-hand det.

$$= \begin{vmatrix} y_1 & y_2 \\ -py_1' - qy_1 & -py_2' - qy_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' & -py_2' \end{vmatrix}$$

(adding $q \cdot$ (1^{st} row) to 2^{nd} doesn't change value of the determinant)

$$= -p\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW.$$

(* you also have to use that y_1, y_2 are solns, i.e., that $y_1'' = -py_1' - qy_1$, $y_2'' = -py_2' - qy_2$).

b) From part (a), if $p(x) = 0$, then $\frac{dW}{dx} = 0$, so $W(y_1, y_2) = c$.

c) $y'' + k^2y = 0$ Here $p = 0$

$W(\cos kx, \sin kx)$

$$= \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix}$$

$= k(\cos^2 kx + \sin^2 kx)$

$= k$, a constant.

2B-1

a) $y_2 = ue^x$

$\underline{x\ -2}|$ $y_2' = u'e^x + ue^x$

$y_2'' = u''e^x + 2u'e^x + ue^x$

Multiply second row by -2 and add:

$y_2'' - 2y_2' + y_2 = u''e^x$ (all other terms cancel out)

If y_2 is a soln to the ODE, the left-hand side must be 0. Therefore we must have

$u''e^x = 0$

so $u'' = 0$,

$\therefore u = ax + b$

and $\therefore y_2 = (ax+b)e^x$

Any of these for which $a \neq 0$ gives a second solution — for ex., $\boxed{y_2 = xe^x}$.

b) From II/7a, $\frac{dW}{dx} = -pW = 2W$

$\therefore W(y_1, y_2) = ce^{2x}$, $c \neq 0$

But $W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$

Equating these two expressions for W,

$e^x(y_2' - y_2) = ce^{2x}$

$\therefore y_2' - y_2 = ce^x$

(c can have any $\neq 0$ value)

Solving this ODE gives (it's a linear equation)

$y_2 = e^x(x + c_1)$ as a family of second solutions.

c) $y_2 = e^x \int \frac{1}{e^{2x}} e^{-\int -2dx} dx$

$= e^x \int 1 \cdot dx = e^x(x + c)$

[more generally: $e^{\int 2dx} = e^{2x + c_1}$

$\therefore y_2 = e^x \int (e^{c_1}) dx$ put $c_2 = e^{c_1}$

$= e^x(c_2 x + c)$]

d) All the solutions are the same — the most general form is

$y_2 = e^x(c_1 x + c_2)$, with $c_1 \neq 0$

(if $c_1 = 0$, we just get y_1 back)

2B-2

$W(y_1, y_2) = \begin{vmatrix} e^x & e^x(ax+b) \\ e^x & e^x(ax+b) + ae^x \end{vmatrix}$

$= ae^{2x}$, $\neq 0$ if $a \neq 0$.

[This shows it for the special equation only].

In general:

$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$

$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} dx$

$\therefore y_2' = y_1' \int \frac{1}{y_1^2} e^{-\int p dx} + y_1 \cdot \frac{1}{y_1^2} e^{-\int p dx}$

$= y_1' y_2 / y_1 + \frac{1}{y_1} e^{-\int p dx}$

$\therefore W(y_1, y_2) = y_1' y_2 + e^{-\int p dx} - y_1' y_2$

$= e^{-\int p dx} \neq 0$

[Note that this same formula for the Wronskian follows from II/7a].

2B-3

let $y_2 = x \cdot u$, so that

$y_2' = u + xu'$, $y_2'' = 2u' + xu''$.

Substituting into $x^2 y'' + 2xy' - 2y = 0$ gives after cancellation and dividing by x^2:

$xu'' + 4u' = 0$ Put $v = u'$.

$x \frac{dv}{dx} + 4v = 0$ or $\boxed{\frac{dv}{v} = \frac{-4dx}{x}}$

Solving, $v = \frac{c}{x^4}$, or $u' = \frac{c}{x^4}$

$\therefore u = \frac{c}{-3x^3} + c_0 = \frac{c_1}{x^3} + c_0$

$\therefore \boxed{y_2 = \frac{c_1}{x^2} + c_0 x}$, a second sol'n (if $c_1 \neq 0$)

[can also use the general formula given in II/8c]

2B-4

Using the general formula [II/8c]:

Find: $e^{-\int p dx}$ $\int p\, dx = \int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$

$= \frac{1}{1-x^2}$

$\therefore \int \frac{1}{x^2} e^{-\int p dx} = \int \frac{dx}{x^2(1-x^2)}$

we do this by partial fractions → (cont'd)

2B-4 (cont'd)

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)} = \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\therefore \int \frac{dx}{x^2(1-x^2)} = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) = -\frac{1}{x} + \frac{1}{2}\ln\frac{(1+x)}{(1-x)}$$

$$\therefore \ y_2 = \underbrace{y_1}_{=x}\int \frac{1}{y_1^2} e^{-\int p\,dx} = \boxed{-1 + \frac{x}{2}\ln\frac{1+x}{1-x}}$$

The general solution is now $c_1 y_1 + c_2 y_2$

or $\boxed{c_1 x + c_2\left(-1 + \frac{x}{2}\ln\frac{1+x}{1-x}\right)}$

2C-1

a) Char eq'n: $\lambda^2 - 3\lambda + 2 = 0$ or $(\lambda-1)(\lambda-2) = 0$

roots: $\lambda = 1, 2$

$$\therefore \boxed{y = c_1 e^x + c_2 e^{2x}}$$

b) Char eq'n: $r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$\therefore \ y = c_1 e^x + c_2 e^{-3x}$. Put in initial conditions:

$y(0)=1 \Rightarrow c_1 + c_2 = 1$

$y'(0)=1 \Rightarrow c_1 - 3c_2 = -1$

solve for c_1, c_2

$c_1 = 1/2, \quad c_2 = 1/2$

$$\therefore \boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{-3x}}$$

c) Char. eqn $r^2 + 2r + 2 = 0$

By quad. formula: $r = -1 \pm i$

$\therefore \ y = e^{-x}(c_1 \cos x + c_2 \sin x)$

[using as y_1, y_2 the real + imaginary parts of the cx. soln $y = e^{(-1+i)x} = e^{-x}(\cos x + i \sin x)$]

2C-1

d) Char. eqn: $r^2 - 2r + 5 = 0$

By quad. fmla: $r = 1 \pm 2i$

Gen'l soln: $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$

Putting in initial cond'ns (you'll have to find y' first!)

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 1 \Rightarrow c_1 + 2c_2 = -1, \quad \therefore c_2 = -1$

so $y = e^x(\cos 2x - \sin 2x)$

e) Char. eqn: $r^2 - 4r + 4 = 0$

or $(r-2)^2 = 0$; $r = 2$ double root

$\therefore \ y = e^{2x}(c_1 x + c_2)$

is the general solution. Put in initial conditions:

$y(0) = 1 \Rightarrow c_2 = 1$

$y'(0) = 1 \Rightarrow 2c_2 + c_1 = 1, \quad \therefore c_1 = -1$

so sol'n is: $y = (1-x)e^{2x}$

2C-2

$$W = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(a\sin bx + b\cos bx) \end{vmatrix}$$

$$= \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ -e^{ax}b\sin bx & e^{ax}(b\cos bx) \end{vmatrix},$$

(by subtracting $a\cdot$(1st row) from 2nd row);

$$= e^{2ax}(b\cos^2 bx + b\sin^2 bx) = e^{2ax}\cdot b$$

$\neq 0$ if $\boxed{b \neq 0}$ (no restriction on a)

2C-3

Char. eqn: $r^2 + cr + 4 = 0$

roots: $r = \dfrac{-c \pm \sqrt{c^2 - 16}}{2}$

a) has oscillatory solns $\Leftrightarrow$ r is complex (so soln has sin + cos terms);

$\Leftrightarrow c^2 - 16 < 0$, or $\boxed{-4 < c < 4}$

b) if the solutions oscillate, above shows that $r = -\frac{c}{2} \pm i\beta \quad (\beta \neq 0)$

and solns are $y = e^{-\frac{cx}{2}}(c_1 \cos\beta x + c_2 \sin\beta x)$.

Damped oscillations $\Leftrightarrow$ $c > 0$ (so $y \to 0$ as $t \to \infty$)

$\therefore \boxed{0 < c < 4}$ is condition.

2C-4

a) [Use y' for $\frac{dy}{dx}$, $\dot{y}$ for $\frac{dy}{dt}$.]

We have $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$ $\qquad x = e^t,\ \frac{dx}{dt} = e^t,\ \frac{dt}{dx} = e^{-t}$

$$\therefore\ y' = \dot{y}\,e^{-t}$$

$$y'' = \frac{d}{dt}(\dot{y}e^{-t})\cdot\frac{dt}{dx}$$

$$= (\ddot{y}e^{-t} - \dot{y}e^{-t})e^{-t}$$

$$= (\ddot{y} - \dot{y})e^{-2t}$$

Substituting into the ODE:

$x^2y'' + pxy' + qy = 0$ becomes

$$(\ddot{y} - \dot{y}) + p\dot{y} + qy = 0$$

b) $p = q = 1$, so we get $\ddot{y} + y = 0$, whose solution are $y = c_1\cos t + c_2\sin t$

$\left.\begin{array}{l} x = e^t \\ \therefore t = \ln x \end{array}\right\}$ gives $y = c_1\cos(\ln x) + c_2\sin(\ln x)$

2C-5

Char. eqn is $\boxed{Mr^2 + cr + k = 0}$

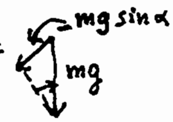
For critical damping, it should have two equal roots; by quadratic formula

$$r = \frac{-c \pm \sqrt{c^2 - 4Mk}}{2M}, \qquad \therefore \boxed{c^2 - 4mk = 0} \text{ is condition}$$

(when $c^2 - 4mk < 0$, get oscillations).

2C-6

Force triangle



$\vec{F} = ma$ becomes:

$$-mg\sin\alpha - mc\frac{d\alpha}{dt} = m\ell\frac{d^2\alpha}{dt^2}$$

(grav.) (air res.)

$$\therefore \boxed{\ddot{\alpha} + \frac{c}{\ell}\dot{\alpha} + \frac{g}{\ell}\sin\alpha = 0}$$

If α small, $\sin\alpha \approx \alpha$

If undamped, $c = 0$, get approx. $\boxed{\ddot{\alpha} + \frac{g}{\ell}\alpha = 0}$ $\left[\text{char eqn is } r^2 + g/\ell = 0\right]$

$\therefore$ Solns are $y = c_1\cos\sqrt{\frac{g}{\ell}}\,t + c_2\sin\left(\sqrt{\frac{g}{\ell}}\,t\right)$

The period $= \dfrac{2\pi}{\sqrt{g/\ell}} = 2\pi\sqrt{\dfrac{\ell}{g}}$

(so as length increases, so does the period; on the moon (less g), it swings slower (bigger period))

2C-7

a) $a + bx + ce^x$ $\quad$ b) $a\cos 2x + b\sin 2x$

c) $ax\cos 2x + bx\sin 2x$

d) ax^2e^x (1 is a double root of the char. eqn)

e) $ae^{-x} + bxe^{2x}$ (2 is a root of char. eqn)

f) $(ax^3 + bx^2)e^{3x}$ (3 is double root of char. eqn)

2C-8

b) $y_h = a_1\cos 2x + a_2\sin 2x$

To find y_p, use undet. coefficients:

$y_p = c_1\cos x + c_2\sin x$ $\quad$ $\lfloor \times 4$ (mult. factor) and add: LHS is by hypothesis: $y_p'' + 4y_p = 2\cos x$

$\therefore\ y_p'' = -c_1\cos x - c_2\sin x$

$2\cos x = 3c_1\cos x + 3c_2\sin x$

$\therefore c_1 = 2/3 \quad c_2 = 0$

So $\boxed{y = a_1\cos 2x + a_2\sin 2x + \frac{2}{3}\cos x}$

$y(0) = 0 \Rightarrow a_1 + 2/3 = 0$ $\qquad \therefore \boxed{\begin{array}{l} a_1 = -2/3 \\ a_2 = 1/2 \end{array}}$

$y'(0) = 1 \Rightarrow 2a_2 = 1$

2C-8

a) $y_h = a_1e^x + a_2e^{5x}$, as usual.

Try $y_p = cxe^x$ $\quad \lfloor\times 5$ } multiplic'n factors

$\therefore\ y_p' = ce^x(x+1)$ $\quad \lfloor\times -6$

$y_p'' = ce^x(x+2)$ $\quad$ then add:

$$e^x = e^x(-6c + 2c) + xe^x(\underbrace{5c - 6c + c}_{0})$$

$\therefore -4c = 1$

$c = -1/4$

$$\boxed{y = a_1e^x + a_2e^{5x} - \frac{1}{4}xe^x}$$

c) Char eqn: $r^2 + r + 1 = 0$, $r = \frac{-1 \pm \sqrt{-3}}{2}$

$\therefore\ y_h = e^{-x/2}\left(a_1\cos\frac{\sqrt{3}}{2}x + a_2\sin\frac{\sqrt{3}}{2}x\right)$

Try $y_p = c_1xe^x + c_2e^x$

$y_p' = c_1e^x(x+1) + c_2e^x$ $\qquad$ Add the eqns:

$y_p'' = c_1e^x(x+2) + c_2e^x$

$2xe^x = 3c_1xe^x + (3c_1 + 3c_2)e^x$

$\therefore c_1 = 2/3, \quad c_2 = -2/3$

$$y = e^{-x/2}\left(a_1\cos\frac{\sqrt{3}}{2}x + a_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{2}{3}e^x(x-1)$$

2C-8

d) $y_h = a_1 e^x + a_2 e^{-x}$

Try: $y_p = c_1 x^2 + c_2 x + c_3$ $\underline{\cdot -1}$

$y_p'' = 2c_1$ Add:

$x^2 = -c_1 x^2 + c_2 x + 2c_1 - c_3$

$\therefore c_1 = -1,\ c_2 = 0,\ 2c_1 - c_3 = 0$
$c_3 = -2$

$$\boxed{y = a_1 e^x + a_2 e^{-x} - x^2 - 2}$$

$y(0) = 0 \Rightarrow a_1 + a_2 - 2 = 0$

$y'(0) = -1 \Rightarrow a_1 - a_2 = -1$

solving, $\boxed{a = 1/2,\ a_2 = 3/2}$

$\therefore y = \frac{1}{2}e^x + \frac{3}{2}e^{-x} - x^2 - 2$

2C-9

a) Write the ODE as $\boxed{Ly = r}$, where L is the linear operator

$L = D^2 + pD + q$

• By hypothesis,

$Ly_1 = r_1$ ← (i.e., y_1 is a solution to $Ly = r_1$)

$Ly_2 = r_2$ ← (similarly)

Adding, $L(y_1 + y_2) = r_1 + r_2$

(using the linearity of L: $L(y_1+y_2) = Ly_1 + Ly_2$)

$\therefore y_1 + y_2$ solves $Ly = r_1 + r_2$.

b) First consider $y'' + 2y' + 2y = 2x$

Try $y_1 = c_1 x + c_2$ $\underline{\cdot 2}$

$y_1' = c_1$ $\underline{\cdot 2}$

$y_1'' = 0$ Add

$2x = 2c_1 x + (2c_2 + 2c_1)$

$\therefore c_1 = 1,\ c_2 = -1$ $\boxed{y_1 = x - 1}$

Then: $y'' + 2y' + 2y = \cos x$

Try $y_2 = a_1 \cos x + a_2 \sin x$ $\underline{\cdot 2}$

$y_2' = -a_1 \sin x + a_2 \cos x$ $\underline{\cdot 2}$

$y_2'' = -a_1 \cos x - a_2 \sin x$ Add

$\cos x = \cos x\,(2a_1 + 2a_2 - a_1) + \sin x\,(2a_2 - 2a_1 - a_2)$

$\therefore \left.\begin{aligned} a_1 + 2a_2 &= 1 \\ -2a_1 + a_2 &= 0 \end{aligned}\right\}$ $\therefore a_2 = 2/5,\ a_1 = 1/5$ $y_2 = \frac{1}{5}\cos x + \frac{2}{5}\sin x$

2C-10

a) $R = 0,\ E = 0$

Eqn is $Lq'' + \frac{q}{C} = 0$ or $q'' + \frac{q}{LC} = 0$

Solving as usual,

$$\boxed{q = c_1 \cos\frac{1}{\sqrt{LC}}t + c_2 \sin\frac{1}{\sqrt{LC}}t}$$

Period is $2\pi\sqrt{LC}$ ($= 2\pi/\text{frequency}$)

frequency $= 1/\sqrt{LC}$

b) Char. eqn is $Lr^2 + Rr + \frac{1}{C} = 0$

roots: $r = \dfrac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$

oscillates if $\boxed{R^2 - \frac{4L}{C} < 0}$

c) $Li'' + \frac{i}{C} = \omega E_0 \cos \omega t$

Solns of homog. eqn are

$i = a_1 \cos\frac{1}{\sqrt{LC}}t + a_2 \sin\frac{1}{\sqrt{LC}}t$

The particular soln i_p will have form $c_1 \cos\omega t + c_2 \sin\omega t$ unless $\omega = \frac{1}{\sqrt{LC}}$, in which case it will be $c_1 t\cos\omega t + c_2 t \sin\omega t$, which gets large as $t \to \infty$.

Thus if $\boxed{\omega \approx \frac{1}{\sqrt{LC}}}$, solns will be large in amplitude

∴ this is ω_0

———o———

> The advantage of this method (divide and conquer!) is that we don't have to assume
> $y_p = d_1 x + d_2 + d_3 \cos x + d_4 \sin x$,
> which would give 4 equations in 4 unknowns to solve ...

Using part (a), the particular solution to

$y'' + 2y' + 2y = 2x + \cos x$

is

$y = y_1 + y_2 = x - 1 + \frac{1}{5}\cos x + \frac{2}{5}\cos x$

2D-1

a) $y_h = \underset{y_1}{c_1\cos x} + \underset{y_2}{c_2\sin x}$, as usual.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $y_p = u_1 y_1 + u_2 y_2$

The equations for variation of pars. are:

$$u_1'\cos x + u_2'\sin x = 0$$
$$u_1'(-\sin x) + u_2'\cos x = \tan x$$

Either by elimination, or by Cramer's rule, we get as sol'n: (the denom. is $W(y_1, y_2)$)

$$u_1' = \frac{-y_2 f(x)}{W(y_1,y_2)} = -\sin x\tan x = \cos x - \sec x$$ (so it can be integrated)

$$u_2' = \frac{y_1 f(x)}{W(y_1,y_2)} = \cos x\tan x = \sin x$$

∴ $u_1 = \sin x - \ln|\sec x + \tan x|$ (from tables)

$u_2 = -\cos x$

∴ $y_p = (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x\sin x$

i.e., $\boxed{y_p = -\cos x\,(\ln|\sec x + \tan x|)}$

b) Two indept soln's of the assoc. homog. eqn are: $y_1 = e^x$, $y_2 = e^{-3x}$ (method as usual)

$W(y_1, y_2) = -4e^{-2x}$ $\left(= \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix}\right)$

$y_p = u_1 y_1 + u_2 y_2$ ⊛

The eqn's for variation of parameters are:

$$u_1'e^x + u_2'e^{-3x} = 0$$
$$u_1'e^x - u_2'(3e^{-3x}) = e^{-x}$$ ← $f(x)$, (from orig. eqn.)

Solve them by elimination, or by Cramer's rule; following the latter, we get as sol'n

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{1}{4}e^{-2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x\cdot e^{-x}}{-4e^{-2x}} = -\frac{1}{4}e^{2x}$$

∴ $u_1 = -\frac{1}{8}e^{-2x}$, $u_2 = -\frac{1}{8}e^{2x}$

and so $y_p = -\frac{1}{8}e^{-2x}\cdot e^x - \frac{1}{8}e^{2x}\cdot e^{-3x}$, by ⊛;

or: $\boxed{y_p = -\frac{1}{4}e^{-x}}$

c) Two indept solns of the assoc. homog. eqn are: $y_1 = \cos 2x$, $y_2 = \sin 2x$ (by the usual method)

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $y_p = u_1 y_1 + u_2 y_2$

Then
$$\begin{cases} u_1'\cos 2x + u_2'\sin 2x = 0 \\ u_1'(-2\sin 2x) + u_2'(2\cos 2x) = \sec^2 2x \end{cases}$$

are the eqn's for the method of var. of pars.

Solving them by elimination, or by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin 2x}{2\cos^2 2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos 2x}{2\cos^2 2x} = \frac{\sec 2x}{2}$$

Integrating,

$$u_1 = -\frac{1}{4}\cdot\frac{1}{\cos 2x}$$

$$u_2 = \frac{1}{4}\ln|\sec 2x + \tan 2x|$$

∴ $\boxed{y_p = -\frac{1}{4} + \frac{1}{4}\ln|\sec x + \tan x|\cdot\sin 2x}$

2D-2

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{x}$, after some calculation.

$$y_p = u_1 y_1 + u_2 y_2$$

Equations for method of var. of pars. are:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = \frac{\cos x}{\sqrt{x}}$$ ← (note: the ODE must be written $y'' + \frac{1}{x}y' + (-)y = \frac{\cos x}{\sqrt{x}}$) ↖ $f(x)$

Solving these by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \cos^2 x$$

$$u_2' = \frac{y_1 f(x)}{W} = -\sin x\cos x$$

∴ $u_1 = \frac{x}{2} + \frac{\sin 2x}{4}$, $u_2 = \frac{\cos 2x}{4}$

and so (using identities):

$$y_p = \frac{\sin x}{\sqrt{x}}\left(\frac{x}{2} + \frac{2\sin x\cos x}{4}\right) + \frac{\cos x}{\sqrt{x}}\left(\frac{\cos^2 x - \sin^2 x}{4}\right)$$

so $y_p = \frac{x\sin x}{2\sqrt{x}} + \frac{1}{4}\frac{\cos x}{\sqrt{x}}$

(The term $\frac{1}{4}\frac{\cos x}{\sqrt{x}}$ is part of the general soln $y = y_p + c_1\frac{\cos x}{\sqrt{x}} + c_2\frac{\sin x}{\sqrt{x}}$; so it can be omitted:

$\boxed{y_p = \frac{\sqrt{x}\sin x}{2}}$ is the best answer)

2D-3

a) Let y_1, y_2 be indep't solutions of the associated homogeneous equation.

$$y_p = u_1 y_1 + u_2 y_2, \qquad W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x), & y_2'(x) \end{vmatrix}$$

and the eqn's for the method of var. of pars. are

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

Solving by Cramer's rule gives

$$u_1' = \frac{-y_2(x) f(x)}{W[y_1(x), y_2(x)]}, \qquad u_2' = \frac{y_1(x) f(x)}{W[y_1(x), y_2(x)]}$$

so that (use definite integrals so as to get a definite function)

$$u_1(x) = \int_a^x \frac{-y_2(t) f(t)\,dt}{W[y_1(t), y_2(t)]}, \qquad u_2(x) = \int_a^x \frac{y_1(t) f(t)\,dt}{W[y_1(t), y_2(t)]}$$

Thus: $\quad y_p(x) = u_1(x)\cdot y_1(x) + u_2(x) y_2(x)$ —

we can put $y_1(x)$ and $y_2(x)$ inside the integral sign because they are "constants" — the integration is with respect to t, not x; then we can add the integrands. The result is:

$$y_p = \int_a^x \frac{-y_1(x)\, y_2(t) + y_2(x) y_1(t)}{W[y_1(t), y_2(t)]} \cdot f(t)\,dt$$

or

$$y_p = \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} f(t)\,dt}{W(y_1(t), y_2(t))}$$

b) The arbitrary constants of integration — call them a_1 and a_2, — will change u_1 and u_2 by an additive constant:

$$u_1 + a_1, \qquad u_2 + a_2$$

leading to the particular soln:

$$y_p = (u_1 + a_1) y_1 + (u_2 + a_2) y_2$$

$$(*) \qquad y_p = \boxed{u_1 y_1 + u_2 y_2} + a_1 y_1 + a_2 y_2$$

The boxed part is the particular solution of part (a); the part added on is in the general soln y_h to the associated homog. eqn, hence the particular soln (*) is just as good a particular soln as the previous one.

2D-4

It depends on the ODE form — (it must be linear!)

<u>Undetermined coefficients</u> requires

1. The ODE is linear, with <u>constant coefficients</u>
2. The inhomogeneous term $f(x)$ has a special form: a sum of terms of the form

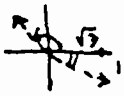
$$(\text{polynomial}) \cdot e^{ax} \cdot \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}.$$

can be 1 — a can be 0 — b can be 0

If the coeffs. are not constant, or $f(x)$ is not of the above form, you must use variation of parameters to find y_p.

<u>Drawback</u>: you must be able to find y_1, y_2 <u>first</u> — i.e., solve the assoc. homog. eq'n.

(Note that finding y_p by undet. coeffs. does <u>not</u> require you to solve for y_1, y_2 first (unless you are unlucky and $f(x)$ is a soln of the assoc. homog. eqn — but you can always test this without solving the eqn.)

2E-1 

$-1+i = \sqrt{2}\,e^{i3\pi/4}$

$\sqrt{3}-i = 2e^{-i\pi/6}$

2E-2 $\dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{(1+i)(1-i)} = \dfrac{-2i}{2} = -i$

Other way:

$1-i = \sqrt{2}\,e^{-i\pi/4}$

$1+i = \sqrt{2}\,e^{i\pi/4}$

$\therefore \dfrac{1-i}{1+i} = \dfrac{\sqrt{2}}{\sqrt{2}} \cdot e^{i(-\pi/4-\pi/4)}$

$= e^{-i\pi/2} = -i$

2E-4 $z = a+bi, \quad w = c+di$

$zw = (ac-bd) + i(ad+bc)$

$\therefore \overline{zw} = (ac-bd) - i(ad+bc)$

$\bar{z}\,\bar{w} = (a-bi)(c-di),$

$= (ac-bd) - i(ad+bc)$

2E-7 a) $(1-i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 + (-i)^4$

$= 1-6+1 + i(-4+4) = \boxed{-4}$

By DeMoivre:

$1-i = \sqrt{2}\,e^{-i\pi/4}$

$(1-i)^4 = (\sqrt{2})^4 e^{-i\pi} = 4\cdot(-1)$

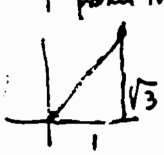
$= -4.$

b) $(1+i\sqrt{3})^3 = 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3$

$= 1 + 3i\sqrt{3} + 3\cdot -3 + i^3 3\sqrt{3}$

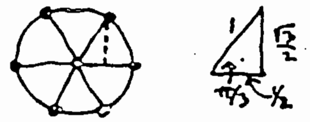
$= -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8$

By polar form:



$1+i\sqrt{3} = 2e^{i\pi/3}$

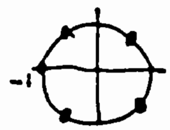
$(1+i\sqrt{3})^3 = 8e^{i\pi} = -8$



2E-9 The sixth roots of 1 are $e^{\frac{i\pi}{3}k}$ where $k = 0, 1, 2, \ldots, 5$

get $\therefore$ $1, -1, \dfrac{\pm 1 \pm i\sqrt{3}}{2}$.

2E-10 $\sqrt[4]{-16} = 2\cdot\sqrt[4]{-1}$.



The 4th roots of -1 are on the picture: $\dfrac{\pm 1 \pm i}{\sqrt{2}}$

$\therefore \sqrt{2}\cdot(\pm 1 \pm i)$ are the roots of $x^4 + 16 = 0$.

2E-14 $\sin^4 x = \left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^4$; by bin. thm, this

$= \frac{1}{16}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right)$

$= \frac{1}{16}\left(e^{4ix} + e^{-4ix}\right) - \frac{4}{16}\left(e^{2ix} + e^{-2ix}\right) + \frac{6}{16}\cdot 1$

$= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$.

Since $\sin^4 x$ is an <u>even</u> function, the answer should not contain the <u>odd</u> functions $\sin 4x$, $\sin 2x$.

2E-15 $e^{(2+i)x} = e^{2x}(\cos x + i\sin x)$

So $e^{2x}\sin x = \operatorname{Im} e^{(2+i)x}$.

$\int e^{(2+i)x}\,dx = \dfrac{1}{2+i} e^{(2+i)x}$; $\dfrac{1}{2+i}\cdot\dfrac{2-i}{2-i} = \dfrac{2-i}{5}$;

$= \dfrac{2-i}{5}\left(e^{2x}\cos x + ie^{2x}\sin x\right)$

We want just the imaginary part:

$\therefore \int e^{2x}\sin x\,dx = e^{2x}\left(\frac{2}{5}\sin x - \frac{1}{5}\cos x\right)$

2E-16 $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

since $\cos(-x) = \cos x$ + $\sin(-x) = -\sin x$,

Adding: $\dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

Subtract: $\dfrac{e^{ix} - e^{-ix}}{2i} = \sin x$

$\boxed{2F-1}$

a) $D^2+2D+2=0$ has roots $-1\pm i$

$\therefore\ y = e^{2x}(c_1+c_2x+c_3x^2) + e^{-x}(c_4\cos x + c_5 \sin x)$

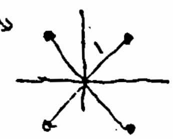
b) $D^8-2D^4+1 = (D^4-1)^2 = [(D^2-1)(D^2+1)]^2$

$= (D-1)^2(D+1)^2(D^2+1)^2$

$\therefore\ y = e^{x}(c_1+c_2x) + e^{-x}(c_3+c_4x) + \cos x\,(c_5+c_6x) + \sin x\,(c_7+c_8x)$

c) Characteristic eq'n is $\boxed{z^4+1=0}$

Roots are $\sqrt[4]{-1}$



$\frac{1\pm i}{\sqrt2}$ and $\frac{-1\pm i}{\sqrt2}$

letting $a = 1/\sqrt2$, get $\therefore$

$y = e^{ax}(c_1\cos ax + c_2 \sin ax) + e^{-ax}(c_3 \cos ax + c_4 \sin ax)$

d) Char. eq'n is $\boxed{z^4-8z^2+16=0}$

which factors as

$(z^2-4)^2$ or $(z+2)^2(z-2)^2$

$\therefore$ has double roots at $2, -2$

so

$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$

e) $y = c_1e^x + c_2e^{-x} + e^{x/2}(c_3\cos\frac{\sqrt3}{2}x + c_4 \sin\frac{\sqrt3}{2}x) + e^{-x/2}(c_5\cos\frac{\sqrt3}{2}x + c_6\sin\frac{\sqrt3}{2}x)$

[using roots as given in soln to 2E-9]

f) $y = e^{\sqrt2 x}(c_1\cos\sqrt2 x + c_2 \sin\sqrt2 x) + e^{-\sqrt2 x}(c_3\cos\sqrt2 x + c_4 \sin\sqrt2 x)$

$\boxed{2F-2}$

$y'''' - 16y = 0$

characteristic equation $\boxed{z^4-16=0}$

roots: $2,\ 2i,\ -2,\ -2i$

(one real root is 2, so the others are all of the form $2\cdot\sqrt[4]{1}$, where $\sqrt[4]{1} = 1, i, -1, -i$)

From roots, general sol'n is

$y = c_1e^{2x} + c_2e^{-2x} + c_3\sin 2x + c_4\cos 2x$

Putting in conditions:

$\boxed{c_1=0}$ since $|y(x)|<K$ for all $x>0$

$(\because |c_1e^{2x}|\to\infty$ unless $c_1=0)$

$y(0)=0 \Rightarrow c_2+c_4=0 \quad \therefore c_4=-c_2$

$y'(0)=0 \Rightarrow -2c_2+2c_3=0 \quad \therefore c_3=c_2$

$\therefore$ sol'n is – so far –

$\boxed{y = c_2(e^{-2x} + \sin 2x - \cos 2x)}$

Finally

$y(\pi)=1 \Rightarrow c_2(e^{-2\pi}-1) = 1$,

$\therefore\ \boxed{c_2 = \frac{1}{e^{-2\pi}-1}}$.

$\boxed{2F-3}$

a) $\boxed{z^3-z^2+2z-2=0}$ is char. eq'n.

1 is a root, $\therefore z-1$ is factor

get $(z-1)(z^2+2)$ roots: $1, i\sqrt2, -i\sqrt2$

$y = c_1e^x + c_2\cos\sqrt2 x + c_3\sin\sqrt2 x$

b) $\boxed{z^3+z^2-2=0} = (z-1)(z^2+2z+2)$

roots $1, -1\pm i$

$\therefore\ y = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x$

c) $(D^3-2D-4) = (D-2)(D^2+2D+2)$

$\therefore\ y = c_1e^{2x} + e^{-x}(c_2\cos x + c_3\sin x)$ roots are $-1\pm i$

d) ↙

d): $\boxed{2F-3}$ $x^4+2x^2+4=0$; $\therefore\ x^2 = \frac{-2\pm\sqrt{2^2-4\cdot4}}{2} = -1\pm\sqrt{-3} = -1\pm\sqrt3\, i$;

changing to polar representation: $= 2e^{2\pi/3\, i},\ 2e^{4\pi/3\, i}$

$\therefore\ x = \sqrt2 e^{\pi/3\, i},\ \sqrt2 e^{4\pi/3\, i}$ (square roots of the first ↗)

$= \sqrt2 e^{2\pi/3\, i},\ \sqrt2 e^{5\pi/3\, i}$ (" " " " others ↗)

$\boxed{\text{↖ and ↙ are conjugates}}$

Using therefore just $\sqrt2 e^{\pi/3\, i}$ and $\sqrt2 e^{\frac{2\pi i}{3}}$, :

$\sqrt2 e^{\pi/3\, i} = \sqrt2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \sqrt2(\frac12 + i\frac{\sqrt3}{2})$; similarly, get $\sqrt2(-\frac12 + i\frac{\sqrt3}{2})$

leading to: $y = e^{\sqrt2/2\, x}(c_1\cos\frac{\sqrt6}{2}x + c_2\sin\frac{\sqrt6}{2}x) + e^{-\sqrt2/2\, x}(c_3\cos\frac{\sqrt6}{2}x + c_4\sin\frac{\sqrt6}{2}x)$

2F-4

$$x_1'' + 2x_1 - x_2 = 0$$
$$x_2'' + x_2 - x_1 = 0$$

Eliminate x_1 by solving for x_1:

$$x_1 = x_2'' + x_2$$

substitute into first equation:

$$(x_2'' + x_2)'' + 2(x_2'' + x_2) - x_2 = 0$$

or $x_2'''' + 3x_2'' + x_2 = 0$

char. eqn: $\boxed{z^4 + 3z^2 + 1 = 0}$

as quadratic eqn in z^2: solve, get

$z^2 = \frac{-3 \pm \sqrt{5}}{2}$: both nos. are real, + negative; call them $-a^2, -b^2$

$\therefore z^2 = -a^2, \quad z^2 = -b^2$

$z = \pm ia, \quad z = \pm ib$

so $x_2 = c_1 \cos at + c_2 \sin at + c_3 \cos bt + c_4 \sin bt$

2F-5

$$D^2 e^{2x} \cos x = e^{2x}(D+2)^2 \cos x = e^{2x}(D^2 + 4D + 4)\cos x = e^{2x}(3\cos x - 4\sin x)$$

2F-6

a) By (12) in notes, (see Example 2)

$$y_p = \frac{4}{1^2+1} e^x = 2e^x$$

b) $(D^3 + D^2 - D + 2)y = 2e^{ix}$

$$\therefore y_p = \frac{2e^{ix}}{i^3 + i^2 - i + 2} = \frac{2(1+2i)}{(1-2i)(1+2i)} e^{ix}$$

$\therefore y_p = \frac{2+4i}{5}(\cos x + i \sin x)$ $\therefore \mathrm{Re}(y_p) = \boxed{\frac{2}{5}\cos x - \frac{4}{5}\sin x}$

c) $(D^2 - 2D + 4)y = e^{(1+i)x}$

$(1+i)^2 - 2(1+i) + 4 = 2$ $\quad \therefore y_p = \frac{e^{(1+i)x}}{2}$

$\mathrm{Re}(y_p) = \boxed{\frac{1}{2} e^x \cos x}$

d) $D^2 - 6D + 9 = (D-3)^2$ $\quad \therefore y_p = cx^2 e^{3x}$

$(D-3)^2 y_p = c e^{3x} D^2 x^2$ (by exp-shift rule)

$= 2c e^{3x} = e^{3x}$ (from the ODE)

$\therefore c = 1/2, \quad y_p = \frac{1}{2} x^2 e^{3x}$

2F-7

$(D+a)e^{-ax}u = e^{-ax}Du = f(x)$

$\therefore Du = e^{ax} f(x), \quad u = \int e^{ax} f(x)\,dx$

$y_p = e^{-ax} \int e^{ax} f(x)\,dx$

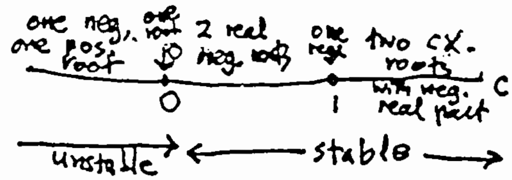
2G-1 $\quad y'' + 2y' + cy = 0$

Char. eqn is:

$$r^2 + 2r + c = 0$$

By quadratic formula:

$$\text{roots} = \frac{-2 \pm \sqrt{4 - 4c}}{2} = -1 \pm \sqrt{1-c}.$$



2G-2

$$r^2 + \frac{b}{a}r + \frac{c}{a} = (r - r_1)(r - r_2)$$

$\therefore \frac{b}{a} = -(r_1 + r_2)$ $\quad (*)$

$\frac{c}{a} = r_1 r_2$

Real case: $r_1, r_2 < 0 \Rightarrow b/a > 0$, $c/a > 0$

$\therefore a, b, c$ have same sign.

Complex case:

$r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\quad \alpha < 0 \Rightarrow \frac{b}{a} = -2\alpha > 0$

by $(*)$ $\frac{c}{a} = \alpha^2 + \beta^2 > 0$.

2G-3

Assume $a, b, c > 0$ (if not, multiply ODE through by -1).

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are the roots.

If roots are real, $\frac{-b - \sqrt{\ }}{2a} < 0$

and $-b + \sqrt{b^2 - 4ac} < 0$, therefore (since $b^2 - 4ac < b^2$).

If roots are complex, $\frac{-b}{2a} < 0$

$\therefore$ in both cases, the char. roots have negative real part.

$\boxed{2H-1}$ $y'' - k^2 y = 0$, $y(0)=0$, $y'(0)=1$

$y_c = c_1 e^{kx} + c_2 e^{-kx}$

Soln to IVP is

$$W(t) = \frac{e^{kx} - e^{-kx}}{2k} = \frac{\sinh kx}{k}$$

$\boxed{2H-3a}$ By Example 2 (p. 2),

$w(x) = xe^{-2x}$

Therefore

$$y(x) = \int_0^x \underbrace{(x-t)e^{-2(x-t)}}_{w(x-t)} \cdot \underbrace{e^{-2t}}_{f(t)} dt$$

$$= e^{-2x} \int_0^x (x-t)\,dt$$

$$= e^{-2x}\left(xt - \frac{t^2}{2}\right)\Big|_0^x = \boxed{\frac{x^2}{2} e^{-2x}}$$

By undetermined coeffs, since $y_c = e^{-2x}(c_1 + c_2 x)$, try $cx^2 e^{-2x}$

$$(D+2)^2 c e^{-2x} x^2 = c e^{-2x} D^2 x^2$$
$$= c e^{-2x} \cdot 2$$

From the ODE,

$$c e^{-2x} \cdot 2 = e^{-2x}, \quad \boxed{c = \frac{1}{2}} \checkmark$$

$\boxed{2H-4}$ a) By Leibniz:

$$\phi'(x) = \frac{d}{dx} \int_0^x (2x+3t)^2 dt =$$

$$= (5x)^2 + \int_0^x 2\cdot(2x+3t)\cdot 2\,dt$$

$$= (5x)^2 + 4\left(2xt + \frac{3t^2}{2}\right)\Big|_0^x = (5x)^2 + 14x^2$$
$$= \boxed{39x^2}$$

b) Directly:

$$\phi(x) = \frac{1}{9}(2x+3t)^3\Big|_0^x = \frac{1}{9}\left((5x)^3 - (2x)^3\right)$$
$$= 13x^3$$

So $\phi'(x) = 39x^2$. $\checkmark$