Recitation 2, February 4, 2010

Direction fields, integral curves, isoclines, separatrices, funnels

Solution suggestions

1. Draw a big axis system and plot some isoclines, especially the nullcline. Use them to illustrate the direction field. Using the direction field, plot a few solutions.

Having done on paper, use a Java-capable browser, go to http://math.mit.edu/mathlets/mathlets and select the "Isoclines" applet. Set the equation to be $y' = x - 2y$. Move the m slider on the right to generate different isoclines (yellow lines), and click on the axis system to plot the solutions (blue curves). Compare the results with what you drew earlier.

2. One of the integral curves seems to be a straight line. Is this true? What straight line is it? (i.e., for what m and b is $y = mx + b$ a solution?)

Yes, one of the curves is a straight line. The function $y = \frac{1}{2}x - \frac{1}{4}$ is a solution to the equation, since $y' = \frac{1}{2} = x - 2(\frac{1}{2}x - \frac{1}{4})$. Therefore, $m = \frac{1}{2}$ and $b = -\frac{1}{4}$.

3. In general — for the general differential equation $y' = F(x, y)$ — if a straight line is an integral curve, how is it related to the isoclines of the equation? What happens in our example?

If a straight line is an integral curve, then it is part of the isocline for its slope. In our example, the line in question 2 forms the entire $y' = \frac{1}{2}$ isocline.

4. It seems that all the solutions become asymptotic as $x \to \infty$. We will see later that this is true, but for now, explain why solutions get trapped between parallel lines of fixed slope.

Since the isoclines have the form $m = x - 2y$, the isoclines are lines of slope $\frac{1}{2}$. When $m > \frac{1}{2}$, any solution through a point on this isocline must pass from the region below the line to the region above it. In particular, no solution can cross in the other direction. Similarly, when $m < \frac{1}{2}$, any solution through a point on the $y' = m$ isocline must pass from the region above the line to the region below it. Since the $y' = m$ isocline has the form $y = \frac{1}{2}x - \frac{m}{2}$, the isoclines for $m > \frac{1}{2}$ lie below the line $y = \frac{1}{2}x - \frac{1}{4}$, while the isoclines for $m < \frac{1}{2}$ lie above it. This traps solutions between any two isoclines that surround the line $y = \frac{1}{2}x - \frac{1}{4}$.

5. Where are the critical points of the solutions of $y' = x - 2y$? How many critical points can a single solution have? For what values of y_0 does the solution y with $y(0) = y_0$ have a critical point? When there is one, is it a

minimum or a maximum? You can see an answer to this from your picture. Can you also use the second derivative test to be sure?

The critical points of the solutions lie on the $y' = 0$ isocline. Since this isocline is a line of slope $\frac{1}{2}$, any solution through a point on it must pass from the region above it to the region below it. Therefore, these solutions pass through the nullcline exactly once, so they have exactly one critical point. Solutions that lie below the line $y = \frac{1}{2}x - \frac{1}{4}$ always have slope greater than $\frac{1}{2}$, since the isoclines of lower slope lie above that line. These solutions, along with the line $y = \frac{1}{2}x - \frac{1}{4}$, have no critical points.

A solution will have a critical point if and only if it lies above the line $y = \frac{1}{2}x - \frac{1}{4}$, so the admissible values of y_0 are those greater than $-\frac{1}{4}$.

All of the critical points are minima.

To use the second derivative test, we take the derivative of both sides of the equation: $y'' = 1 - 2y'$. Since $y' = 0$ at the critical point, we have $y'' = 1$. The second derivative is positive, so at least the critical point is a local minimum. However, for any solution that lies above the line $y = \frac{1}{2}x - \frac{1}{4}$, before it crosses the line $x - 2y = 0$, y decreases since $y' < 0$, and afte the crossing, y increases since $y' > 0$. Hence, the critical point is actually a global minimum

6. For another example, take $y' = y^2 - x^2$. (This is on the `Isoclines` Mathlet.) Again, make a BIG picture of some isoclines and use them to sketch the direction field, and then sketch a few solutions.

Set the equation to be $y' = y^2 - x^2$ in the applet, and follow the same procedure as in question 1.

7. A "separatrix" is a solution such that solutions above it have a fate (as x increases) entirely different from solutions below it. The equation $y' = y^2 - x^2$ exhibits a separatrix. Sketch it and describe the differing behaviors of solutions above it and below it.

You can draw something resembling the separatrix in the applet by clicking near the top right of the picture (e.g., click near $(3.70, 3.84)$). Solutions above it rapidly increase without bound. Solutions below it approach the lower right asymptote of the $y' = -1$ isocline, which lies on the hyperbola $y^2 = x^2 - 1$.

8. The equation $y' = y^2 - x^2$ also exhibits a "funnel," where solutions get trapped as x increases, and many solutions are asymptotic to each other. Explain this using a couple of isoclines. There is a function with a simple formula (not a solution to the equation, though) which all these trapped solutions get near to as x gets large. What is it?

Solutions below the separatrix approach the $y' = -1$ isocline. This is a hyperbola, whose lower right asymptote is the line $y = -x + 1$.

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