Recitation 7, February 25, 2010

Solutions to second order ODEs

Solution suggestions

1. Check that both $x = \cos(\omega t)$ and $x = \sin(\omega t)$ satisfy the *second order* linear differential equation

$$\ddot{x} + \omega^2 x = 0$$

This equation is called the *harmonic oscillator*.

If $x = \cos(\omega t)$, then $\dot{x} = -\omega\sin(\omega t)$ and $\ddot{x} = -\omega^2\cos(\omega t) = -\omega^2 x$. If $x = \sin(\omega t)$, then $\dot{x} = \omega\cos(\omega t)$ and $\ddot{x} = -\omega^2\sin(\omega t) = -\omega^2 x$.

2. In fact, check that *any* sinusoidal function with circular frequency ω, $A\cos(\omega t - \phi)$, satisfies the equation $\ddot{x} + \omega^2 x = 0$.

If $x = A\cos(\omega t - \phi)$, then $\dot{x} = -A\omega\sin(\omega t - \phi)$, and $\ddot{x} = -A\omega^2\cos(\omega t - \phi) = -\omega^2 x$.

3. Among the functions $x(t) = A\cos(\omega t - \phi)$, which have $x(0) = 0$? Doesn't this contradict the uniqueness theorem for differential equations?

$x(0) = A\cos\phi$. When $A = 0$, then $x(t) = 0$ for every t; when $A \neq 0$, $x(0) = 0$ implies $\cos\phi = 0$, and hence ϕ can be any odd multiple of $\pi/2$, i.e., $\phi = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ This does not contradict the uniqueness theorem, because the uniqueness theorem as stated only applies to first order equations.

4. Given numbers x_0 and $\dot{x}_0$, can you find a solution to $\ddot{x} + \omega^2 x = 0$ for which $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$? How many such solutions are there?

Suppose $x(t) = a\cos(\omega t) + b\sin(\omega t)$. $x(0) = a$, so $a = x_0$. $x'(0) = -a\omega\sin 0 + b\omega\cos 0 = b\omega = \dot{x}_0$. Then $b = \dot{x}_0/\omega$. The solution is then $x = x_0\cos(\omega t) + \dot{x}_0\sin(\omega t)/\omega$. There is only one such solution.

5. Suppose that r is a (perhaps complex) constant such that e^{rt} is a solution to $\ddot{x} + kx = 0$. What does r have to be?

Let $x = e^{rt}$. Then $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$, and $\ddot{x} + kx = (r^2 + k)e^{rt}$. We want this to be zero, but since e^{rt} is never zero, it must be $r^2 = -k$. Suppose $k \geq 0$, then r must have the form $\pm i\sqrt{k}$.

6. Find a solution x_1 to $\ddot{x} - a^2 x = 0$ [note the sign!] such that $x_1(0) = 1$ and $\dot{x}_1(0) = 0$. Find another solution x_2 such that $x_2(0) = 0$ and $\dot{x}_2(0) = 1$.

We use a similar approach as we studied $\ddot{x} + \omega^2 x = 0$. First, notice that both $x(t) = e^{at}$ and $x(t) = e^{-at}$ are solutions to $\ddot{x} - a^2 x = 0$. Then for any constants c_1 and c_2, $x(t) = c_1 e^{at} + c_2 e^{-at}$ are also solutions to $\ddot{x} - a^2 x = 0$. Moreover, $x(0) = c_1 + c_2$, and $\dot{x}(0) = a(c_1 - c_2)$. Assuming $a \neq 0$, to get $x_1(t)$, we need $c_1 + c_2 = 1$ and $a(c_1 - c_2) = 0$, which implies $c_1 = c_2 = 1/2$. So $x_1(t) = \cosh(at)$. For $x_2(t)$, set $c_1 + c_2 = 0$ and $a(c_1 - c_2) = 1$, so $c_1 = -c_2 = \frac{1}{2a}$ and $x_2(t) = \sinh(at)/a$.

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