18.03 Recitation 21, April 27, 2010

First order linear systems

Vocabulary/Concepts: system of differential equations; linear, time-independent, homogeneous; matrix, matrix multiplication; solution, trajectory, phase portrait; companion matrix.

1. Practice in matrix multiplication: Compute the following products:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x & u \\ y & v \end{bmatrix}.$$

2. Multiplying by a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ sends a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to another vector $A\begin{bmatrix} x \\ y \end{bmatrix}$. This operation can be visualized by thinking about where it sends the square with corners $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{i}+\mathbf{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For each of the following matrices A, draw segments connecting the dots $\mathbf{0}$, $A\mathbf{i}$, $A(\mathbf{i}+\mathbf{j})$, $A\mathbf{j}$, $\mathbf{0}$, and invent verbal description or name for the operation.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

3. What is the companion matrix A of the second order equation $\ddot{x} + 2\dot{x} + 2x = 0$? Find two independent real solutions of this second order equation. Let $x_1(t)$ denote the solution with initial condition $x_1(0) = 0$, $\dot{x}_1(0) = 1$. Find it, and then write down the corresponding solution $\mathbf{u_1}(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix}$ of the equation $\dot{\mathbf{u}} = A\mathbf{u}$. What is $\mathbf{u_1}(0)$? Sketch the graphs of $x_1(t)$ and of $\dot{x}_1(t)$, and sketch the trajectory of the solution $\mathbf{u_1}(t)$. Compare these pictures.

Sketch a few more trajectories to fill out the phase portrait. In particular sketch the trajectory of $\mathbf{u_2}(t)$ with $\mathbf{u_2}(0) = \mathbf{i}$.

When trajectories of this companion equation cross the x axis, at what angle do they cross it?

4. Let $a + bi$ be a general complex number. There is a matrix A such that if $(a + bi)(x + yi) = (v + wi)$ then

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$$

Find it. What is it for $a + bi = 2$? For $a + bi = i$? For $a + bi = 1 + i$? Draw the parallelograms discussed in **(2)** for these matrices.

18.03 Differential Equations
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