18.034, Honors Differential Equations
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Lecture 4: Existence & Uniqueness, Part II

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1. Cauchy sequences and complete metric spaces.

A metric space is complete if it "has no holes". What does this mean? A metric space "has a hole" if there is a sequence that "should converse", but such that there is no limit. The sequences which "should converse" are Cauchy sequences.

Def'n: Let (X,d) be a metric space. A sequence of elements in X, $(P_i)_{i=0,1,\ldots}$, is a Cauchy sequences if for every $\varepsilon > 0$, there is an integer $M \geq 0$ such that for all pairs of integers $m,n \geq M$, the distance $d(P_m, P_n) < \varepsilon$.

Thm [$\approx$Heine-Borel thm] For the Euclidean metric space ($\mathbb{R}^n$, ∂^{Eucl}), sequence (P_i) is convergent if and only if it is Cauchy.

This property of ($\mathbb{R}^n$, ∂^{Eucl}) has a name.

Def'n: A metric space (X, ∂) is complete if for every sequence of elements in X, (P_i), the sequence is convergent if and only if it is Cauchy.

Exercise: Prove that for any metric space, if (P_i) is convergent, then (P_i) is Cauchy.

2 Cauchy test

Let [a,b] be a bounded interval in $\mathbb{R}$, and let $C([a,b],\mathbb{R})$ be the metric space of continuous real valued functions on [a,b] with the metric $\partial\ (y_2, y_1) =$ maximum value of $|y_2(t) - y_1(t)|$ on [a,b]. The main theorem about this metric space is the following.

Thm [Cauchy test = Thm A.21]: Let $(y_i(t))$ be a sequence in $C([a,b],\mathbb{R})$. If (y_i) is Cauchy, then it converses, i.e. $|C([a,\ b],\mathbb{R}),\ \partial|$ is a complete metric space.

Pf: Suppose (y_i) is Cauchy. Then for every t in [a,b], the sequence of real numbers $(y_i,(t))_{i=0,1,\ldots}$, is a Cauchy sequence. Because $\mathbb{R}$ is complete, $(y_i,(t))$ converges to some real number. Call this number y(t).

The claim is that, for every $\varepsilon > 0$, there is an integer $N \geq 0$ such that for all $n \geq N$, $\max\{(y_n(t) - y(t)\} < \varepsilon$. To prove this, observe by the Cauchy condition that there is $N \geq 0$ such that for all $m,n \geq N$, $\max\max\left\{\ |y_m(t) - y_n(t)|\right\} < \frac{\varepsilon}{2}$.

Now, for each t, because $(y_i,t))$ converges to y(t), there exists $M = M(t) \geq 0$ such that for all $m \geq M$, $|y_m(t) - y(t)| < \frac{\varepsilon}{2}$. But then, for every $n \geq N$ (N doesn't depend on t),

$$|y_n(t) - y(t)| = |(y_n(t) - y_m(t)) + (y_m(t) - y(t))|$$
$$|(y_n(t) - y_m(t)) + (y_m(t) - y(t))|$$
$$< \varepsilon/2 \qquad + \varepsilon/2 \qquad = \varepsilon.$$

(Cauchy condition) $\quad$ (y_m(t) converses to y(t))

This proves the claim.

If we knew that y(t) is continuous, it would follow that (y_i) converses to y. So let's prove y(t) is continuous. Let t be any element in [a,b]. By the claim, there exists $N \geq 0$ such that for all $n \geq N$, $\max\{(y_n(\delta) - y(t)\} < \frac{\varepsilon}{3}$. Because $y_n(t)$ is continuous, there exists z >0 such that

$|y_n(s) - y_n(t)| < \varepsilon/3$ whenever $|t - s| < z$,

Then, whenever $|t - s| < z$

$$|y(s) - y(t)| \leq |y(s) - y_n(s)| + |y_n(s) - y_n(t)| + |y_n(t) - y(t)|$$

(by the triangle inequality)

$$\leq \underset{\left(b/c \max\{|y_n - y|\} < \varepsilon/3\right)}{\varepsilon/3} + \underset{\left(b/c\ y_n(t) cts\right)}{\varepsilon/3} + \underset{\left(b/c \max\{|y_n - y|\} < \varepsilon/3\right)}{\varepsilon/3}$$

$$= \varepsilon$$

So y(t) is continuous. Therefore y(t) is an element of $C([a,b], \mathbb{R})$, and the sequence ($y_i(t)$) converges to y(t).

<u>Variation</u>: The metric space $B_{\|\bullet\|}(y_0, b)$ is complete (defined as in lecture 3).

<u>Proof</u>: The only new step is to prove that the limit y(t) has graph contained in $[t_0, t_0 + C] \times [y_0 - b, y_0 + b]$. But for each t, y(t) is the limit of ($y_i(t)$). Because $y_i(t)$ is a sequence in the closed interval $[y_0 - b, y_0 + b]$, its limit is also in this interval. So $y_0 - b \leq y(t) \leq y_0 + b$, i.e. y(t) is in $B_{\|\bullet\|}(y_0, b)$.

(Contd. on next page…)

3. Thm [Contraction mapping fixed point theorem, Part II]: Let (X, ∂) be a complete metric space. Let ε be a number with $0 \le \varepsilon < 1$. Let T be an ε-contraction mapping on (X, ∂). Then there exists a unique fixed point of T.

Proof: We already proved there is at most one point. The real content is that there exists a fixed point.

Let p_0 be any point of (X, ∂). Define $p_1 = T(p_0)$, define $p_2 = T(p_1)$, etc. In other words, define a sequence $(p_i)_{i=0,1,\ldots}$ of elements in X inductively by $T(p_i) = p_{i+1}$.

Denote by D the distance $D = \partial(p_1, p_0)$. The claim is that for all i=0,1,..., $\partial(p_{i+1}, p_i) \le \varepsilon^i \bullet D$.
This is proved by induction on i, the base case being done. If $\partial(p_{i+1}, p_i) \le \varepsilon^i \bullet D$, then

$$\partial(p_{i+2}, p_{i+1}) = \partial(T(p_{i+1}), T(p_i))\ (T(p_{i+1}), T(p_i)) \le \varepsilon \bullet \partial(p_{i+1}, p_i) \quad \text{(b/c T is ε-contracting)}$$
$$\le \varepsilon \bullet \varepsilon^i \bullet D \quad \text{(by hypothesis)}$$
$$= \varepsilon^{i+21} \bullet D$$

This proves the induction step, thus $\partial(p_{i+1}, p_i) \le \varepsilon^i \bullet D$ for all i.

The claim is that (p_i) is a Cauchy sequence. Let $\varepsilon' > 0$ be any number. Let N be an integer such that $\frac{\varepsilon^N . D}{1-\varepsilon} < \varepsilon'$, i.e.

$$N > \log \frac{(1-\varepsilon)\varepsilon'}{D} \Big/ \log(\varepsilon).$$

Then for all $m \ge n \ge N$,

$$\begin{aligned}\partial(p_m, p_n) &\le \partial(p_{n+1}, p_n) + \partial(p_{n+2}, p_{n-1}) + \ldots + \partial(p_m, p_{m-1}) \\ &\le \varepsilon^n \bullet D + \varepsilon^{n+1} + 1 \bullet D + \varepsilon^{n+2} + 2 \bullet D + \ldots + \varepsilon^{m-1} \bullet D \\ &\le \varepsilon^n \bullet D + \varepsilon^{n+1} + 1 \bullet D + \varepsilon^{n+2} + 2 \bullet D + \ldots = \varepsilon^n \bullet D \bullet \frac{1}{1-\varepsilon}.\end{aligned}$$

Since $\frac{\varepsilon^N . D}{1-\varepsilon} < \varepsilon'$ for $n \ge N$. $\partial(p_m, p_n) < \varepsilon'$.
So (p_i) is a Cauchy sequence.

Because (X, ∂) is a complete metric space, the X Cauchy sequence (p_i) converses to an element p of X. Let $\varepsilon' > 0$ be a number. There exists N such that for all $m \ge n \ge N$, $\partial(p_m, p_n) < \varepsilon/3$.
Also, there is $n \ge N$ such that $\partial(p, p_n) < \varepsilon/3$.
Thus, $\partial(p, T(p)) \le \partial(p, p_n) + \partial(p_n, p_{n+1}) + \partial(p_{n+1}, T(p))$.
But $\partial(p_{n+1}, T(p)) = \partial(T(p_n), T(p)) < \partial(p_n, p)$.
So $\partial(p_{n+1}, T(p)) \le 2 \bullet \partial(p, p_n) + \partial(p_n + p_{n+1}) < 2 \bullet \varepsilon/3 + \varepsilon/3 = \varepsilon$.
Since $\partial(p, T(p))\ (p, T(p))$ is less than ε for all $\varepsilon > 0$, $\partial(p, T(p)) = 0$. Therefore $p = T(p)$, i.e. p is a fixed point of T.

4. Existence of a solution to the IVP. Let R, D, f, M, L, a, b and c be as in Lecture 3.

Thm: There exists a differentiable function y(t) defined on $[t_0, t_{0+c}]$ such that
(1) $y(t_0) = y_0$

(2) $y'(t) = f(t, y(t))$
(3) $|y(t) - y_0| \leq b$

Proof: As proved in Lecture 3, on the metric space $B_{\|\bullet\|}(y_0, b)$ of continuous functions y(t) on $[t_0, t_{0+c}]$ such that $|y(t) - y_0| \leq b$, the mapping $T(y) = z$,

$$z(t) := y_0 + \int_{t_0}^{t} f(s, y(s))ds .$$

is a 1/2 – contraction mapping. By the Cauchy test, $B_{\|\bullet\|}(y_0, b)$ is a complete metric space. By the contraction mapping fixed point theorem, part II, there exists a continuous function y(t) in $B_{\|\bullet\|}(y_0, b)$ such that $y(t) = T(y(t))$, i.e.

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds .$$

By the fundamental theorem of calculus, y(t) is differentiable and $y'(t) = f(s, y(s))$. Moreover,

$$y(t_0) = y_0 + \int_{t_0}^{t_0} f(s, y(s))ds = y_0 + 0 = y_0 .$$

Finally since y(t) is in $B_{\|\bullet\|}(y_0, b)$, $|y(t) - y_0| \leq b$ for all t in $[t_0, t_0+c]$.