18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #6

(modified on April 8, 2009)

Due by Friday, April 10, 2009, by NOON.

1. (Laplace transform of t^r). The *Gamma* function is defined by the integral

$$\Gamma(r+1) = \int_0^\infty e^{-t} t^r \, dt.$$

(a) Show that the improper integral converges for all $r > -1$.

(b) Show that $\Gamma(r+1) = r\Gamma(r)$ for $r > 0$. Show that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

(c) For $r > -1$ show that $\mathcal{L}[t^r] = \Gamma(r+1)/s^{r+1}$, $s > 0$

2. (a) Find the solution of the initial value problem

$$y'' + \omega^2 y = h(t)\sin t - h(t-c)\sin t, \qquad y(0) = y'(0) = 0,$$

where $c > 0$ is a constant and $\omega^2 \neq 1$.

(b) Show that $y(0) = y'(0) = y''(0) = 0$. Show that y and y' are continuous at $t = c$.

(c) Show that $y''(c+) - y''(c-) = -\sin c$, which is 0 if and only if $c = n\pi$ for n an integer. This behavior is explained by that the function $h(t-c)\sin t$ is continuous at c if and only if $\sin c = 0$.

3. Find the Laplace transform of a *full rectified wave** $f(t) = |\sin t|$.

4. Find the solution of the initial value problem

$$y'' + y = A\delta(t-c), \qquad y(0) = a, \quad y'(0) = b,$$

where A, a, b, c are constant and $c > 0$.

(b) Show that $y(t) = 0$ for $t \geq c$ if and only if $y(c) = 0$ and

$$A = \sqrt{a^2 + b^2}, \quad a\sin c \geq b\cos c; \qquad A = -\sqrt{a^2+b^2}, \quad a\sin c \leq b\cos c.$$

To interpret, these amplitudes A and locations of impulse c cancel the oscillation.

5. (The Volterra integral equation). Consider the integral equation

$$y(t) + \int_0^t (t-s)y(s)\,ds = -\frac{1}{4}\sin 2t.$$

(a) Show that the above integral equation is equivalent to the initial value problem

$$y'' + y = \sin 2t, \qquad y(0) = 0, \quad y'(0) = -\frac{1}{2}.$$

(b) Solve the integral equation by using the Laplace transform.

*It describes a direct current.

6. Consider the Bessel equation of order zero

$$ty'' + y' + ty = 0.$$

Note that $t = 0$ is a singular point and thus solutions may become unbounded as $t \to 0$. Nevertheless, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there.

(a) Show that $Y(s) = \mathcal{L}[y](s)$ satisfies $(1 + s^2)Y'(s) + sY(s) = 0$.

(b) Using the binomial series for $(1 + s^2)^{-1/2}$ for $s > 1$ show that

$$y(t) = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2},$$

which is referred to as the *Bessel function* of the first kind of order zero. Show that $y(0) = 1$ and y has finite derivatives for all orders at $t = 0$.