18.034 Honors Differential Equations
Spring 2009

LECTURE 9. SEPARATION AND COMPARISON THEOREMS

Many references encourage the impression that computing the Wronskian of two functions is a good way to determine whether or not they are linearly independent. But, two functions are linearly dependent if one is a multiple of the other; otherwise they are linearly independent. It is fairly easy to see by inspection, without computing the Wronskian.

Nevertheless, the Wronskian has an important application in deriving properties of solutions of

$$y'' + p(t)y' + q(t)y = 0, \tag{9.1}$$

where $p(t), q(t)$ are continuous.

Separation and comparison theorems. To motivate, we consider the *Airy equation*

$$y'' - ty = 0, \qquad t < 0. \tag{9.2}$$

The equation looks like the equation of simple harmonic motion

$$y'' + \omega^2 y = 0$$

except that the frequency keeps increasing with t. More precisely, $-t$ in (9.2) is in the position of the frequency ω^2, so near a given value of $-t$, we expect solutions to behave like $\cos\sqrt{-t}t$ and $\sin\sqrt{-t}t$. They are, of course, not solutions of (9.2), but still, they give us a hint of what to expect. Indeed, the (normalized) pair of solutions of (9.2), called the Airy cosine and sine functions behave as in the figure below.

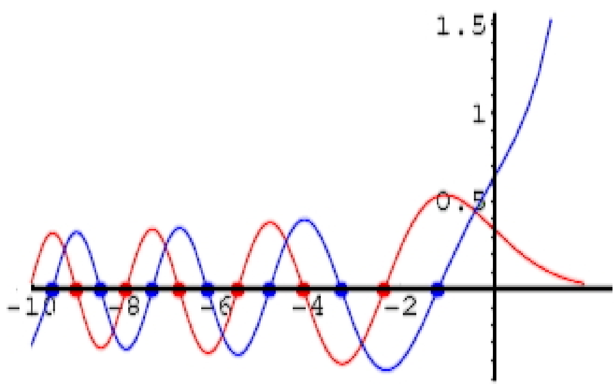

Figure 9.1. Airy cosine and sine.

In fact, all nontrivial solutions of (9.1) have essentially the same number of oscillation (or zeros).

Theorem 9.1 (Sturm Separation Theorem)**.** *If u and v are linearly independent pair of solutions of* (9.1)*, then u must vanish at one point between two successive zeros of v. In other words, the zeros of u and v occur alternately.*

Proof. Let t_1 and t_2 be successive zeros of v. That is, $v(t_1) = v(t_2) = 0$ but $v(t) \neq 0$ on (t_1, t_2). Since u and v are linearly independent, their Wronskian $W(t) := W(u(t), v(t))$ is never zero. Assume $W(t) > 0$ for all t; the case $W(t) < 0$ can be treated similarly. Then,

$$W(t_j) = u(t_j)v'(t_j) > 0, \qquad j = 1, 2.$$

Since t_1 and t_2 are successive zeros of v, it must hold that $v'(t_1)$ and $v'(t_2)$ have opposite signs. Indeed, if v increases at t_1 then it must decrease at t_2, and vice versa. Thus, $u(t_1)$ and $u(t_2)$ must have opposite signs. By the intermediate value theorem, then, u must vanish between t_1 and t_2. This completes the proof. □

A refinement of the above argument can be used a more useful result.

Theorem 9.2 (Sturm Comparison Theorem). *Let*

$$u'' + p(t)u = 0 \quad \textit{and} \quad v'' + q(t)v = 0, \tag{9.3}$$

and let $p(t)$ and $q(t)$ are continuous and $p(t) \geqslant q(t)$. Then u vanishes at least once between any two zeros of v, unless $p(t) \equiv q(t)$ and u is a constant multiple of v.

Proof. Let t_1 and t_2 be successive zeros of v so that $v(t_1) = v(t_2) = 0$ but $v(t) \neq 0$ on (t_1, t_2). Suppose that u failed to vanish on $t \in (t_1, t_2)$. Without loss of generality, we assume that u and v are positive on $t \in (t_1, t_2)$.

We compute

$$W(u, v; t_1) = u(t_1)v'(t_1) \geqslant 0 \quad \text{and} \quad W(u, v; t_2) = u(t_2)v'(t_2) \leqslant 0. \tag{9.4}$$

On the other hand, since $u, v > 0$ and $q \geqslant q$ on the interval (t_1, t_2) , we have

$$\frac{d}{dt}W(u, v; t) = uv'' - u''v = (p - q)uv \geqslant 0 \qquad \textit{on} \quad t \in (t_1, t_2).$$

The second equality uses (9.3). It implies that $W(t)$ is a nonincreasing function of t, and it contradicts with (9.4), unless $p(t) \equiv q(t)$ on $t \in (t_1, t_2)$. In this event, $u(t) = cv(t)$ for some c. This completes the proof. □

Applications. We show that if $q(t) \leqslant 0$ then no nontrivial solution of

$$y'' + p(t)y = 0 \tag{9.5}$$

can have more than one zero. If not, by the Sturm comparison theorem, the solution $v \equiv 1$ of the differential equation $y'' = 0$ would have to have at least one zero between any two zeros of a nontrivial solution of (9.5).

Similarly, if $q(t) \geqslant k^2 > 0$ then any solution of (9.5) must vanish between two successive zeros of any given solution $A\cos(kt - t_1)$ of the differential equation $y'' + k^2u = 0$, hence in any interval of length π/k.

The Bessel equation of order n is

$$t^2y'' + ty' + (t^2 - n^2)y = 0.$$

For $t > 0$ it is commonly written in the normal form as

$$y'' + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right)y = 0.$$

Substituting $y = v/\sqrt{t}$, we write it

$$v'' + \left(1 - \frac{4n^2 - 1}{4t^2}\right)v = 0. \tag{9.6}$$

Note that v vanishes when y does, and vice versa.

We apply the Sturm comparison theorem to (9.2) and $u'' + u = 0$ to state that: each interval of length π in the half-line $t > 0$ contains at least one zero of an solution of the Bessel equation of order zero, and at most one zero of any nontrivial solution of the Bessel equation of order n where $n > 1/2$.

The Sturm comparison theorem extends to "self-adjoint" equations.

Theorem 9.3 (Extended Sturm Comparison Theorem)**.** *Let*

$$(p_1(t)u')' + q_1(t)u = 0, \qquad (p_2(t)v') + q_2(t)v = 0$$

and let $p_1(t) \geqslant p_2(t) > 0$ and $q_2(t) \geqslant q_1(t)$. Then, between any two zeros of a nontrivial solution $u(t)$ of the first equation, there lies at least one zero of any real solution of the second equation, except when $u = cv$ for some c.

Exercise. Show that by substitution $v = e^{\int p dt/2} y$, the equation

$$(p(t)y')' + q(t)y = 0$$

becomes

$$v'' + (q(t) - p^2(t)/4 - p'(t)/2)v = 0.$$