18.034 Honors Differential Equations
Spring 2009

UNIT VI: THE LINEAR SYSTEMS

We study linear systems of n first-order differential equations. They are related to first-order matrix differential equations. When the corresponding matrix is constant, then the eigenvalues and the eigenfunctions of the matrix provide a useful framework to construct the general solution. The fundamental matrix is constructed as the exponential matrix.

LECTURE 25. LINEAR SYSTEMS

A linear system of n differential equations in n unknowns is

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + f_1(t), \\ y_2' &= a_{21}(t)y_1 + \cdots + a_{2n}(t)y_n + f_2(t), \\ &\vdots \qquad \vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + f_n(t). \end{aligned} \tag{25.1}$$

Here and elsewhere, $' = d/dt$. In the matrix notation,

$$\vec{y}\,' = A(t)\vec{y} + \vec{f}(t), \tag{25.2}$$

where $\vec{y} = (y_1(t), \ldots, y_n(t))^T$ and $\vec{f}(t) = (f_1(t), \ldots, f_n(t))^T$ are vector-valued functions defined on an interval $t \in I$ and taking values in $\mathbb{R}^n$, and $A(t) = (a_{ij}(t))_{i,j=1}^n$ is an $n \times n$ matrix-valued function defined on I.

A matrix-valued function $A(t) = (a_{ij}(t))$ is said to be continuous, bounded, or differentiable if each element a_{ij} of $A(t)$ is continuous, bounded and differentiable, respectively. Differentiation and integration are element-wise:

$$\frac{d}{dt}A(t) = \left(\frac{da_{ij}}{dt}(t)\right), \qquad \int A(t)\,dt = \left(\int a_{ij}\,dt\right).$$

Let us define an operator

$$L\vec{y} = \vec{y}\,' - A\vec{y}. \tag{25.3}$$

With this notation, (25.2) is written as $L\vec{y} = \vec{f}(t)$. The domain of the operator L is the space of n-dimensional vector-valued functions differentiable on I.

Exercise. Show that L is linear. That is,

$$L(c_1\vec{y}_1(t) + c_2\vec{y}_2(t)) = c_1L\vec{y}_1(t) + c_2L\vec{y}_2(t).$$

Since L is linear, the fundamental principle of superposition and the principle of the complementary function and other results for linear operators apply.

Existence and Uniqueness result. If $A(t)$ and $\vec{f}(t)$ are continuous and bounded on an interval I containing t_0, then for each $\vec{y}_0 \in \mathbb{R}^n$ the initial value problem

$$\vec{y}\,' = A(t)\vec{y} + \vec{f}(t), \qquad \vec{y}(t_0) = \vec{y}_0$$

has a unique solution in I.

Working assumption. Throughout the note and the following notes, $A(t)$ and $f(t)$ are assumed to be continuous and bounded on an interval $t \in I$.

Linear independence. The *vecotrs* $\vec{y}_1, \dots, \vec{y}_n$ in $\mathbb{R}^n$ are said to be *linearly independent* if

$$c_1\vec{y}_1 + c_2\vec{y}_2 + \cdots + c_n\vec{y}_n = 0 \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

Let $Y = (\vec{y}_1, \dots, \vec{y}_n)$ be an $n \times n$ matrix whose j-th column is $\vec{y}_j$. Then, $\vec{y}_1, \dots, \vec{y}_n$ are linearly independent if and only if the determinant $|Y| \neq 0$. In this case, moreover, these functios form a basis for the linear space $\mathbb{R}^n$.

Similarly, the vector-valued functions $\vec{y}_1(t), \dots, \vec{y}_n(t)$ are said to be *linearly independent* on the interval I if

$$c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + \cdots + c_n\vec{y}_n(t) = 0 \quad \text{on } I \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

This definition is more restrictive than that for vectors. But, if the functions are solutions of linear systems of differential equations, then their linear independence can be characterized in terms of the determinant of a matrix-valued function, analogous to that for vectors.

A *basis of solutions* of

$$\vec{y}\,' = A(t)\vec{y}, \tag{25.4}$$

where $A(t)$ is a continuous $n \times n$ matrix-valued function on I, is a collection of n solutions $\vec{y}_1(t), \dots, \vec{y}_n(t)$ so that every solution can be expressed uniquely as their linear combination

$$\vec{y}(t) = c_1\vec{y}_1(t) + \cdots + c_n\vec{y}_n(t)$$

for constants $c_1, c_2, \dots, c_n$.

Theorem 25.1. *Let $\vec{y}_1(t), \dots, \vec{y}_n(t)$ be n solutions of* (25.4) *on I. Let $Y(t) = (\vec{y}_1(t), \dots, \vec{y}_n(t))$ be the $n \times n$ matrix-valued function, whose j-th column is $y_j(t)$.*

The functions form a basis of solutions if and only if $|Y(t_0)| \neq 0$ at some point $t_0 \in I$. If $|Y(t_0)| = 0$ at a point $t_0 \in I$, then $|Y(t)| = 0$ for all $t \in I$.

If $|Y(t_0)| = 0$ then the column vectors in $Y(t_0)$ are linearly dependent. Thus, we can find constants c_j not all zero such that

$$y(t) = c_1\vec{y}_1(t) + \cdots + c_n\vec{y}_n(t)$$

satisfies $y(t_0) = 0$. Moreover, $y(t)$ is a solution of (25.4). Then, by uniqueness, $y(t) = 0$ for all $t \in I$. That is, the functions are linearly dependent on I.

Notations. $|Y(t)| = \det(\vec{y}_1(t) \cdots \vec{y}_n(t))$ is called the *Wronskian* of $\vec{y}_1(t), \dots, \vec{y}_n(t)$.

If the columns of $Y(t)$ are linearly independent solutions of $\vec{y}\,' = A(t)\vec{y}$ then $Y(t)$ is called the *fundamental matrix* for $\vec{y}\,' = A(t)\vec{y}$. The fundamental matrix yields the general solution, and hence the complete solution, of $\vec{y}\,' = A(t)\vec{y}$. This proves the assertion.

Plane autonomous systems. When $n = 2$, then the system

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2, \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 \end{aligned} \tag{25.5}$$

is called a *plane system*. In the matrix notation, we write it as $\vec{y}\,' = A(t)\vec{y}$, where $\vec{y} = (y_1, y_2)$ and Let $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$.

When a_{ij} are constants, then let $\phi = y_1$. We eliminate y_2 from the system, and

$$(\phi' - a_{11}\phi)' = a_{12}a_{21}\phi + a_{22}(\phi' - a_{11}\phi).$$

Thus, we obtain the *companion* or *secular equation*

$$\phi'' - (\text{tr}\, A)\phi' + (\det A)\phi = 0 \tag{25.6}$$

of (25.5).

Proposition 25.2. *If $(y_1(t), y_2(t))^T$ is a solution of* (25.5)*, then both $y_1(t)$ and $y_2(t)$ are solutions of the secular equation* (25.6)*.*

The proof is easy and it is left as an exercise.

Conversely, the secular equation (25.6) can be used to solve the system (25.5), by finding the roots of the corresponding characteristic polynomial

$$\lambda^2 - (\text{tr}\, A)\lambda + \det A,$$

and subsequently (25.5) can be solved. In this note, we call the quadratic polynomial as the *characteristic polynomial* of A.

Example 25.3. We consider the linear system

$$\begin{aligned} y_1' &= 12y_1 + 5y_2, \\ y_2' &= -6y_1 + y_2, \end{aligned} \qquad A = \begin{pmatrix} 12 & 5 \\ -6 & 1 \end{pmatrix}.$$

Its secular equations is $\phi'' - 13\phi' + 42\phi = 0$, and the solutions are generated by e^{6t} and e^{7t}.

If $y_1(t) = c_1 e^{6t}$, then $5y_2 = y_1' - 12y_1 = -6c_1e^{6t}$. If $y_1(t) = c_2e^{7t}$ then $y_2(t) = -c_2e^{7t}$. Thus, the general solutions of the system is

$$c_1 e^{6t}\begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 e^{7t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Exercise. Find the general solution of

1. $A = \begin{pmatrix} 6 & 1 \\ -1 & 8 \end{pmatrix}$. (ANSWER. $c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} t \\ -t+1 \end{pmatrix}$).
2. $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. (ANSWER. $c_1 e^t \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$.)

Review of linear algebra. We end this note by listing notations and results of linear algebra, useful in the study of linear system of differential equations.

Let $A = (a_{ij})$ be an $n \times n$ matrix. If the rows and the columns of A are interchanged then the resulting matrix is called the *transpose* of A, denoted by A^T. It is immediate that

$$(A^T)^T = A, \qquad (A^T)^{-1} = (A^{-1})^T, \qquad (AB)^T = B^T A^T.$$

A matrix is said to be *symmetric* if $A = A^T$.

We use $|A|$ for the determinant of A. The (i, j)-minor of A, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained when the ith row and the jth column of A are deleted. Then, by definition,

$$|A| = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}.$$

The cofactor of A, denoted by cof A, is a $n \times n$ matrix $(c_{ij})_{i,j=1}^n$, where

$$c_{ij} = (-1)^{i+j}M_{ij}.$$

Then,

$$|A| = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}.$$

Finally, the adjugate of A, denoted by adj A, is the transpose of the cofactor matrix

$$\text{adj}\, A = (\text{cof}\, A)^T, \qquad (\text{adj}\, A)_{ij} = c_{ji}.$$

Then, follows the *Laplaces expansion formula* for determinant

$$|A| = \sum_{j=1}^{n} a_{ij}(\text{cof}\, A)_{ij} = \sum_{i=1}^{n} a_{ij}(\text{cof}\, A)_{ij}$$

for any $1 \leqslant i, j \leqslant n$. That is a determinant can be computed on any row or on any column.

Example 25.4. The adjugate of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is adj $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

We state a useful formula

$$A(\text{adj}\, A) = (\text{adj}\, A)A = (\det A)I. \tag{25.7}$$

The (i, j) entry of $A(\text{adj}\, A)$ is in fact the inner product of the i-th row of A and the i-th row of cof A, and thus (25.7) is simply the Laplace formula.

We define the *characteristic polynomial*

$$p_A(\lambda) = \det(A - \lambda I) \tag{25.8}$$

of the matrix A. Let

$$p_A(\lambda) = (-1)^n \lambda^n + p_1 \lambda^{n-1} + \cdots + p_0.$$

It naturally leads to a matrix polynomial

$$p_A(A) = (-1)^n A^n + p_1 A^{n-1} + \cdots + p_0 I.$$

An important property of the adjugate is

$$\text{adj}\, A = -((-1)^n A^{n-1} + p_1 A^{n-2} + \cdots + p_1 I). \tag{25.9}$$

Formally, adj $A = \dfrac{p(0) - p(A)}{A}$.