18.03SC Practice Problems 14

Frequency Response

In this problem session we will work with a second order mass-spring-dashpot system driven by a force F_{ext} acting directly on the mass: $m\ddot{x} + b\dot{x} + kx = F_{ext}$. So here the input signal is F_{ext} and the system response is x. (Contrast this with the mass-spring-dashpot system examined previously in Session 10.)

We will be interested in a sinusoidal input signal, $F_{ext}(t) = A\cos(\omega t)$, and in the steady state, also sinusoidal, system response, $x_p(t) = gA\cos(\omega t - \phi)$. Here g is the gain of the system and ϕ is the phase lag. Both depend upon ω, and we will examine these dependencies. This setup is illustrated by the Mathlet `Amplitude and Phase: Second Order IV`, which you can have open in parallel to check your intuition as you work through this problem session.

Take $m = 1$, $b = \frac{1}{4}$, and $k = 2$. Also, assume $A = 1$, so that the amplitude of the system response is just the gain. You should be able to select these values on the Mathlet and get an animation of the exact system we will be examining.

1. Write down the equation for the system we are working with. Compute its complex gain $H(\omega)$. (That is, make the complex replacement $F_{cx} = e^{i\omega t}$ for the input signal, and express the exponential system response z_p as a complex multiple of F_{cx}: $z_p = H(\omega)F_{cx}$.)

2. Write down the expression for the gain $g(\omega) = |H(\omega)|$. What is the amplitude of the system response when $\omega = 1$? Use the Mathlet to check your answer.

3. What is the resonant angular frequency ω_r? (Hint: minimize the square of the denominator.) Again, you should be able to check your answer on the Mathlet.

4. On the Mathlet it appears that the phase lag is almost, but not quite, $\frac{\pi}{2}$ at the resonant angular frequency. Verify this computationally: at what frequency is the phase lag exactly one quarter cycle?

5. At what angular frequency is the phase lag equal to $\frac{\pi}{4}$? How about $\frac{3\pi}{4}$? Are your answers supported by the Mathlet?

6. New topic: Find a solution of $\ddot{x} + 3\dot{x} + 2x = te^{-t}$.

7. Explain why a mass-spring-dashpot system driven through the spring is modeled by the equation $m\ddot{x} + b\dot{x} + kx = ky$. Here x measures the displacement of the mass from its position at rest, y measures the displacement of the other end of the spring from its position at rest by the driving force, and $x = y$ when the spring is relaxed.

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