

Domain of $F(s)$

1. Complex s and region of convergence

We will allow s to be complex, using as needed the properties of the complex exponential we learned in unit 1.

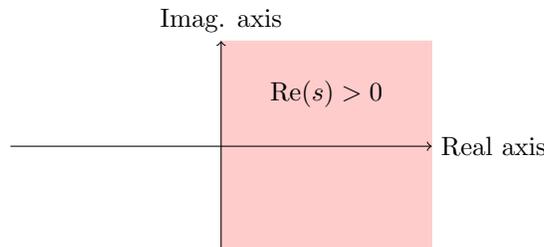
Example 1. In the previous note we saw that $\mathcal{L}(1) = 1/s$, valid for all $s > 0$. Let's recompute $\mathcal{L}(1)(s)$ for complex s . Let $s = \alpha + i\beta$.

$$\begin{aligned}\mathcal{L}(1)(s) &= \int_0^{\infty} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right]_0^T \\ &= \lim_{T \rightarrow \infty} \left. \frac{e^{-\alpha t} (\cos(\beta t) + i \sin(\beta t))}{-s} \right]_0^T\end{aligned}$$

This converges if $\alpha > 0$ and diverges if $\alpha < 0$. Since $\alpha = \operatorname{Re}(s)$ we have

$$\mathcal{L}(1) = 1/s, \quad \text{for } \operatorname{Re}(s) > 0.$$

The region $\operatorname{Re}(s) > 0$ is called the **region of convergence** of the transform. It is a right half-plane.



Region of convergence: right half-plane $\operatorname{Re}(s) > 0$.

Frequency: The Laplace transform variable s can be thought of as complex frequency. It will take us a while to understand this, but we can begin here. Euler's formula says $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ and we call ω the angular frequency. By analogy for any complex number exponent we call s the **complex frequency** in e^{st} . If $s = a + i\omega$ then s is the complex frequency and its imaginary part ω is an actual frequency of a sinusoidal oscillation.

2. Piecewise continuous functions and functions of exponential order

If the integral fails to converge for any s then the function does not have a Laplace transform.

Example. It is easy to see that $f(t) = e^{t^2}$ has no Laplace transform.

The problem is the e^{t^2} grows too fast as t gets large. Fortunately, all of the functions we are interested in do have Laplace transforms valid for $\text{Re}(s) > a$ for some value a .

Functions of Exponential Order

The class of functions that do have Laplace transforms are those of *exponential order*. Fortunately for us, all the functions we use in 18.03 are of this type.

A function is said to be of **exponential order** if there are numbers a and M such that $|f(t)| < Me^{at}$. In this case, we say that f has exponential order a .

Examples. $1, \cos(\omega t), \sin(\omega t), t^n$ all have exponential order 0. e^{at} has exponential order a .

A function $f(t)$ is **piecewise continuous** if it is continuous everywhere except at a finite number of points in any finite interval and if at these points it has a jump discontinuity (i.e. a jump of finite height).

Example. The square wave is piecewise continuous.

Theorem: If $f(t)$ is piecewise continuous and of exponential order a then the Laplace transform $\mathcal{L}f(s)$ converges for all s with $\text{Re}(s) > a$.

Proof: Suppose $\text{Re}(s) > a$ and $|f(t)| < Me^{at}$. Then we can write $s = (a + \alpha) + ib$, where $\alpha > 0$. Then, since $|e^{-ibt}| = 1$,

$$|f(t)e^{-st}| = |f(t)e^{-(a+\alpha)t}e^{-ibt}| = |f(t)e^{-(a+\alpha)t}| < Me^{-\alpha t},$$

Since $\int_0^\infty Me^{-\alpha t} dt$ converges for $\alpha > 0$, the Laplace transform integral also converges.

Domain of $F(s)$: For $f(t)$ we have $F(s) = 1/s$ with region of convergence $\text{Re}(s) > 0$. But, the function $1/s$ is well defined for all $s \neq 0$. The process of extending the domain of $F(s)$ from the region of convergence is called *analytic continuation*. In this class analytic continuation will always consist of extending $F(s)$ to the complex plane minus the zeros of the denominator.

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