18.03SC Practice Problems 24

Step and delta functions

Solution suggestions

1. *Let* $Q(t) = \begin{cases} 0 & \text{for } t < 1 \\ 2t-2 & \text{for } 1 < t < 2 \\ 2t-1 & \text{for } 2 < t < 3 \\ 5 & \text{for } 3 < t \end{cases}$

(a) *Sketch a graph of this function. Is it piecewise smooth?*

The function $Q(t)$ is made up of finitely many nice (differentiable) functions, and so, yes, it is piecewise smooth. The function is graphed in Figure 1 below.

(b) *Find the generalized derivative $q(t) = Q'(t)$, and sketch it.*

We can graph the derivative $q(t) = Q'(t)$ piece by piece, as in Figure 2 on the next page. The derivative has jumps at $t = 1$ and $t = 3$, where the original function has corners, and there is a delta function of magnitude $+1$ at $t = 2$, where $Q(t)$ has a jump discontinuity of height $+1$.

We can also find this derivative algebraically. First we write $Q(t)$ as a generalized function.

$$Q(t) = (2t-2)u(t-1) + u(t-2) + (5-2t+1)u(t-3),$$

and then take the (generalized) derivative and use the product rule to obtain

$$\begin{aligned} q(t) &= Q'(t) \\ &= 2u(t-1) + (2-2)\delta(t-1) + \delta(t-2) + (-2)u(t-3) + (5-2*3+1)\delta(t-3) \\ &= 2(u(t-1) - u(t-3)) + \delta(t-2). \end{aligned}$$

This matches the derivative we found graphically.

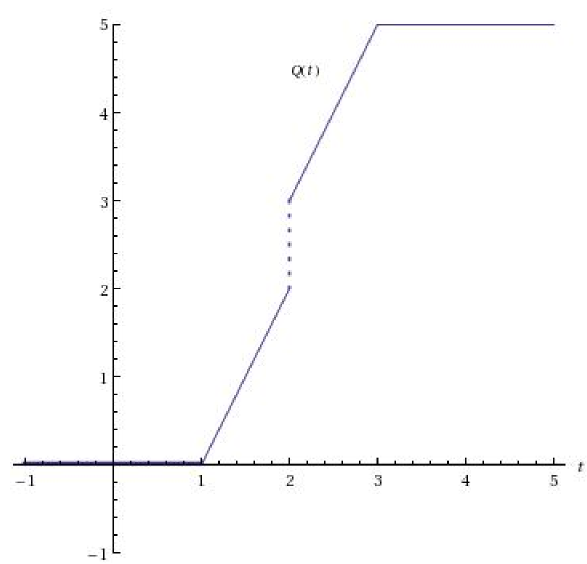

Figure 1: The piecewise-defined function $Q(t)$

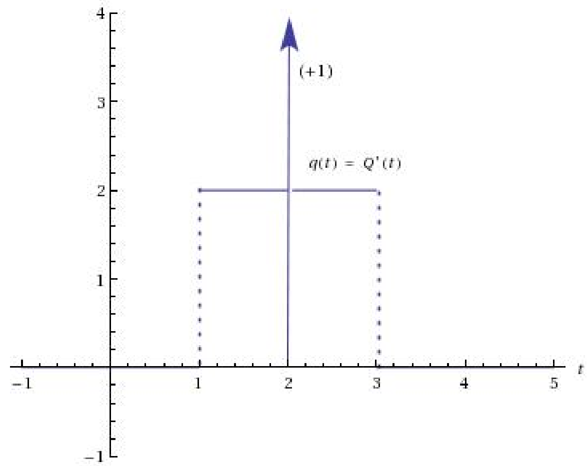

Figure 2: Its generalized derivative $q(t)$

(c) *Describe a scenario which might be modeled by the equation $\dot{x} + kx = q(t)$ (your choice of k) with rest initial conditions.*

Here is a possible scenario for $\dot{x} + kx = q(t)$: The variable we are modeling, x, describes the balance of a bank account (measured in, say, thousands of dollars) which grows over time t (measured in, say, years) through interest at a rate $-k$ (for the DE to model exponential growth we have $k < 0$). The driving function $q = q(t)$ represents the rate at which additional deposits are made into the savings account. Before time $t = 1$, the account balance is zero. Between time $t = 1$ and time $t = 3$, the owner of the account has a job and steadily puts in 2 thousand dollars a year into the bank account - say, by making monthly or weekly deposits. (We are using a continuous approximation here and assuming the contributions are made at a constant rate.) At time $t = 2$, the owner wins a lottery and makes a one-time deposit of a thousand dollars.

(d) *Describe a scenario which might be modeled by the equation $2\ddot{x} + 4\dot{x} + 4x = q(t)$ with rest initial conditions.*

Here is a possible scenario for $2\ddot{x} + 4\dot{x} + 4x = q(t)$: The system $2\ddot{x} + 4\dot{x} + 4x$ describes a mass-spring-dashpot system with constants $m = 2$, $b = 4$, $k = 4$, driven directly by the external force $q(t)$. Before time $t = 1$, the force is zero, the spring and the dashpot are relaxed and the mass is still. Between time $t = 1$ and time $t = 3$, the force is steadily at 2 units. At time $t = 2$, an additional impulse of one unit hits the system through the driving force.