SOLUTION SET VIII FOR 18.075–FALL 2002

5. Boundary-Value Problems and Characteristic-Function Representations

5.1. Introduction.

1. Show that the boundary-value problem $\frac{d^2y}{dx^2} - k^2 y = 0,\ y(0) = y(l) = 0$ cannot have a nontrivial solution for real values of k.

Solution. Let $y = y(x)$ be a solution, i.e., $\begin{cases} y''(x) = k^2\, y(x) \\ y(0) = y(l) = 0 \end{cases}$. We need to show that, if k is real, then $y(x) = 0$ for all x in $[0, l]$.

If $k = 0$, then $y(x) = c_1 x + c_0$ and clearly, $y(0) = y(l) = 0$ implies $c_1 = c_0 = 0$. Hence

$$y(x) = 0 \text{ for } x \text{ in } [0, l].$$

In the case that $k \neq 0$, the general solution is

$$y(x) = c_1 \sinh(kx) + c_0 \cosh(kx).$$

The condition $y(0) = 0$ gives $\underline{c_0 = 0}$. The condition $y(l) = 0$ gives

$$c_1 \sinh(kl) = 0.$$

For nontrivial solutions we must impose $c_1 \neq 0$. Hence, the only possibility of nontrivial solutions is when

$$\sinh(kl) = 0.$$

But the zeros of $\sinh(z) = 0$ are $z = in\pi$, n: integer, i.e., z has to be 0 or pure imaginary! For $z = kl = 0$ we get $k = 0$ which is not acceptable (since $k \neq 0$ by assumption). The other choise of $z = kl = in\pi$ is not acceptable either because k is taken to be real.

It follows that there are no nontrivial solutions of the given boundary-value problem for real k.

2. Determine those values of k for which the partial differential equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ possesses nontrivial solutions of the form $T(x, y) = f(x) \sinh(ky)$ which vanish when $x = 0$ and when $x = l$.

Solution. Let T be of the form $T(x, y) = f(x) \sinh(ky)$. Then,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = (f''(x) + k^2 f(x)) \sinh(ky).$$

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So, if $T(x,y) = f(x)\sinh(ky)$ is a solution with the required properties, then $f(x)$ satisfies

$$\begin{cases} f''(x) + k^2 f(x) = 0 \\ f(0) = f(l) = 0 \end{cases} \tag{1}$$

and $f(x) \neq 0$ for some x in $[0, l]$.

If $k = 0$, then $f(x) = c_1 x + c_2$, and $f(0) = f(l) = 0$ implies $c_1 = c_2 = 0$. So $f(x) \equiv 0$. This shows that the value $k = 0$ is not acceptable.

If $k \neq 0$, then $f(x) = c_1 \cos(kx) + c_2 \sin(kx)$. So $f(0) = f(l) = 0$ implies $c_1 = 0$ and $c_2 \sin(kl) = 0$. Since $f(x) \neq 0$, we must have $c_2 \neq 0$, and hence $kl = m\pi$ for nonzero integer m. So k is determined to be $k = \frac{m\pi}{l}$ where $m = 1, 2, 3 \ldots$.

For any such m, $f(x) = \sin(\frac{m\pi}{l}x)$. Then $f(x)$ cannot vanish identically and satisfies (1). So

$$T(x,y) = \sin\left(\frac{m\pi x}{l}\right) \sinh\left(\frac{m\pi}{l}y\right)$$

is a solution with the required properties. This shows that the set $\{k = \frac{m\pi}{l} \mid m = 1, 2, 3, \ldots\}$ is the set of all characteristic values (eigenvalues) of the given problem.

3. Show that the equation $\frac{d^2y}{dx^2} + \lambda x^2 y = 0$ can possess a nontrivial solution which vanishes at $x = 0$ and at $x = 1$ only if λ is such that $J_{1/4}(\frac{\sqrt{\lambda}}{2}x^2) = 0$ and that, corresponding to such a characteristic number λ_k, any multiple of the function $\varphi_k(x) = \sqrt{x} J_{\frac{1}{4}}(\frac{\sqrt{\lambda_k}}{2}x^2)$ is a solution with the required properties.

Solution. Rewrite the equation as

$$x^2 \frac{d^2y}{dx^2} + \lambda x^4 y = 0.$$

If $\lambda = 0$, then $y(x) = c_1 x + c_0$. Application of the given boundary conditions then gives $c_0 = c_1 = 0$, which is the trivial solution. So, the value $\lambda = 0$ is not admissible.

Therefore, consider $\lambda \neq 0$. According to the theory of **Sec. 4.10**, let $Y = \frac{y}{\sqrt{x}}$, $X = \frac{\sqrt{\lambda}}{2}x^2$. Then the given ODE transforms into the Bessel equation

$$X^2 \frac{d^2Y}{dX^2} + X\frac{dY}{dX} + (X^2 - \frac{1}{16})Y = 0.$$

Hence, $Y = Z_{\frac{1}{4}}(X)$, i.e., the general solution is

$$y = \sqrt{x}\left[C_1 J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2) + C_2 J_{-\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)\right],$$

where C_1 and C_2 are arbitrary constants.

If there is a non-trivial solution such that $y(0) = y(1) = 0$, then there exist $(C_1, C_2) \neq (0,0)$ such that

$$\lim_{x\to 0} \sqrt{x}\left[C_1 J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2) + C_2 J_{-\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)\right] = 0 \tag{2}$$

and

$$(3)\quad \lim_{x\to 1}\sqrt{x}\left[C_1 J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)+C_2 J_{-\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2))\right]=C_1 J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})+C_2 J_{-\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0.$$

From the Frobenius series for the Bessel function, we get the limiting behavior ($p\neq$ integer)

$$J_p(x)\sim\frac{1}{2^p\Gamma(1+p)}\,x^p,\quad J_{-p}(x)\sim\frac{2^p}{\Gamma(1-p)}\,x^{-p},\qquad x\to 0,$$

where $\Gamma(z)$ is the Gamma function. It follows that

$$\lim_{x\to 0}\sqrt{x}J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)=0,$$

while

$$\lim_{x\to 0}\sqrt{x}J_{-\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)=\frac{\sqrt{2}}{\Gamma(\frac{3}{4})}\lambda^{-\frac{1}{8}}.$$

This last term contributes a nonzero value to the left-hand side of (2) unless we eliminate it by choosing $C_2=0$. So (2) dictates that $C_2=0$. Then $C_1\neq 0$, and (3) implies $J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0$.

Thus, if the given boundary-value problem has a nontrivial solution, then $\lambda\neq 0$ and $J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0$.

If $J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0$, from the argument above, we see that $y=C\sqrt{x}J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)$ is a solution for the equation for any C, and it clearly satisfies the given boundary conditions since $J_{\frac{1}{4}}(0)=J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0$.

Now we conclude that the given boundary-value problem has a non-trivial solution if $J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2})=0$ and $\lambda\neq 0$, in which case, any multiple of $\sqrt{x}J_{\frac{1}{4}}(\frac{\sqrt{\lambda}}{2}x^2)$ is a solution of the given boundary-value problem.

4. Show that the equation $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\lambda xy=0$ can possess a nontrivial solution which is *finite* at $x=0$ and which vanishes when $x=a$ only if λ is such that $J_0(\sqrt{\lambda}\cdot a)=0$, and obtain the form of the solution in this case.

Solution. Let $X=\sqrt{\lambda}\cdot x$, then the given ODE becomes

$$X^2\frac{d^2y}{dX^2}+\frac{dy}{dX}+X^2y=0.$$

So the general solution of the given equation is

$$y=Z_0(X)=Z_0(\sqrt{\lambda}\cdot x)=C_1J_0(\sqrt{\lambda}\cdot x)+C_2Y_0(\sqrt{\lambda}\cdot x),$$

where C_1 and C_2 are as yet arbitrary constants.

Then the given boundary-value problem, i.e.,

$$(4)\qquad \begin{cases} x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\lambda xy=0\\ y(0):\ \text{finite}\\ y(a)=0\end{cases}$$

has a non-trivial solution only if there exists $(C_1, C_2) \neq (0,0)$, such that $\lim_{x\to 0}(C_1 J_0(\sqrt{\lambda}\cdot x) + C_2 Y_0(\sqrt{\lambda}\cdot x))$ is a finite number and $C_1 J_0(\sqrt{\lambda}\cdot a) + C_2 Y_0(\sqrt{\lambda}\cdot a) = 0$.

But $\lim_{x\to 0} J_0(\sqrt{\lambda}\cdot x) = J_0(0) = 1$ and $\lim_{x\to 0} Y_0(\sqrt{\lambda}\cdot x) = -\infty$. So $\lim_{x\to 0}(C_1 J_0(\sqrt{\lambda}\cdot x) + C_2 Y_0(\sqrt{\lambda}\cdot x))$ is a finite number only if we eliminate the Y_0 term by choosing $C_2 = 0$. In this case, we must have $C_1 \neq 0$, and hence $J_0(\sqrt{\lambda}\cdot a) = 0$. Now, we conclude that (4) has a nontrivial solution only if $J_0(\sqrt{\lambda}\cdot a) = 0$, and in this case, the argument above shows that the solutions of (4) are of the form $y(x) = C J_0(\sqrt{\lambda}\cdot x)$, where C is arbitrary.