SOLUTION SET IX FOR 18.075–FALL 2004

5. Boundary-Value Problems and Characteristic-Function Representations

5.6. **Orthogonality of Characteristic Functions.**

22. Reduce each of the following differential equations to the standard form $\frac{d}{dx}(p\frac{dy}{dx}) + (q + \lambda r)y = 0$:

(a) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (x + \lambda)y = 0$,
(b) $\frac{d^2y}{dx^2} + \frac{dy}{dx}\cot x + \lambda y = 0$,
(c) $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + (b + \lambda)y = 0$ (a, b constants),
(d) $x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay + \lambda y = 0$ (a, c constants).

Solution. (a)

$$\begin{aligned} x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (x+\lambda)y = 0 \quad &\Leftrightarrow \quad x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 + \lambda x)y = 0 \\ &\Leftrightarrow \quad \frac{d}{dx}(x^2\frac{dy}{dx}) + (x^2 + \lambda x)y = 0 \end{aligned}$$

(b)

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{dy}{dx}\cot x + \lambda y = 0 \quad &\Leftrightarrow \quad \sin x\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx} + \lambda\sin x \cdot y = 0 \\ &\Leftrightarrow \quad \frac{d}{dx}(\sin x\frac{dy}{dx}) + \lambda\sin x \cdot y = 0 \end{aligned}$$

(c)

$$\begin{aligned} \frac{d^2y}{dx^2} + a\frac{dy}{dx} + (b+\lambda)y = 0 \quad &\Leftrightarrow \quad e^{ax}\frac{d^2y}{dx^2} + a \cdot e^{ax}\frac{dy}{dx} + (b \cdot e^{ax} + \lambda \cdot e^{ax})y = 0 \\ &\Leftrightarrow \quad \frac{d}{dx}(e^{ax}\frac{dy}{dx}) + (b \cdot ae^{ax} + \lambda \cdot e^{ax})y = 0 \end{aligned}$$

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(d)

$$\begin{aligned} & x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay+\lambda y=0 \\ \Leftrightarrow\quad & e^{-x}\cdot x^c\frac{d^2y}{dx^2}+e^{-x}\cdot x^c(\frac{c}{x}-1)\frac{dy}{dx}+(\frac{-ae^{-x}x^c}{x}+\frac{\lambda e^{-x}x^c}{x})y=0 \\ \Leftrightarrow\quad & \frac{d}{dx}(e^{-x}x^c\frac{dy}{dx})+(-ae^{-x}x^{c-1}+\lambda e^{-x}x^{c-1})y=0 \end{aligned}$$

23. By considering the characteristic functions of the problem

$$\frac{d^2y}{dx^2}+\mu^2y=0,\ y(0)=0,\ \alpha ly'(l)+y(l)=0,$$

with $\alpha>0$, and using the results of Section 5.6, show that

$$\int_0^l \sin(\mu_1x)\ \sin(\mu_2x)\ dx=0$$

when μ_1 and μ_2 are distinct positive solutions of the equation $\tan(\mu l)+\alpha\mu l=0$.

Solution. The general solution of $\frac{d^2y}{dx^2}+\mu^2y=0$, where $\mu\neq 0$, is $y=c_1\cos\mu x+c_2\sin\mu x$. $\mu^2(\neq 0)$ is a characteristic value only if

(1) $y(0)=0\Rightarrow c_1\cos 0+c_2\sin 0 \ = \ 0,$ and

(2) $\alpha ly'(l)+y(l)=0\ \Rightarrow\ \alpha l[c_1\cos(\mu x)+c_2\sin(\mu x)]'\,|_{x=l}+[c_1\cos(\mu l)+c_2\sin(\mu l)] \ = \ 0$

Equation (1) implies $c_1=0, c_2\neq 0$ and (2) gives $\alpha l\mu\cos(\mu l)+\sin(\mu l)=0$ which is equivalent to

(3) $\tan(\mu l)+\alpha\mu l \ = \ 0$

So the nonzero characteristic values of the given problem are the non-zero solutions of (3). We can easily see that if μ is a solution of (3) then $y=\sin(\mu x)$ is a solution of the given boundary-value problem. Thus, we conclude the set of nonzero characteristic values of the given problem is $\Lambda=\{\mu^2 \text{ real}: \mu>0, \tan(\mu l)+\alpha\mu l=0\}$, and for a μ^2 in Λ the corresponding characteristic function is $\sin(\mu x)$.

Now consider the integration $\int_0^l \sin(\mu_1x)\ \sin(\mu_2x)\,dx$, where $\mu_1\neq\mu_2,\ \mu_1,\mu_2\geq 0,\ \tan(\mu_1l)+a\mu_1l=\tan(\mu_2l)+a\mu_2l=0$. Clearly, if one of $\mu_1,\mu_2=0$, then $\int_0^l \sin(\mu_1x)\ \sin(\mu_2x)\ dx=\int_0^l 0\ dx=0$. If both $\mu_1,\mu_2\neq 0$ then $\sin(\mu_1x)$ and $\sin(\mu_2x)$ are characteristic functions of the given problem, corresponding to different characteristic values. Note that in our case, we have $r(x)\equiv 1$, so $\int_0^l \sin(\mu_1x)\ \sin(\mu_2x)\ dx=\int_0^1 r(x)\ \sin(\mu_1x)\ \sin(\mu_2x)\ dx=0$.

24. By considering the characteristic functions of the problem

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\mu^2xy=0,\ y(0) \text{ finite},\ y(l)=0,$$

and using the results of Section 5.6, show that

$$\int_0^l xJ_0(\mu_1 x)J_0(\mu_2 x)\ dx = 0$$

when μ_1 and μ_2 are distinct positive roots of the equation $J_0(\mu l) = 0$.

Solution. Let $X = \mu x$, then $X^2\frac{d^2y}{dX_2} + X\frac{dy}{dx} + X^2 y = 0$. So the general solution of the given equation is $y = Z_0(X) = Z_0(\mu x) = c_1 J_0(\mu x) + c_2 Y_0(\mu x)$. Then $\mu^2 (\neq 0)$ is a characteristic value of the given boundary-value problem only if

$$\lim_{x\to 0}[c_1 J_0(\mu x) + c_2 Y_0(\mu x)] \text{ is finite and } c_1 J_0(\mu l) + c_2 Y_0(\mu l) = 0.$$

Since $\lim_{x\to 0} J_0(\mu x) = 1$ and $\lim_{x\to 0} Y_0(\mu x) = -\infty$, the finiteness of the limit

$$\lim_{x\to 0}[c_1 J_0(\mu x) + c_2 Y_0(\mu x)] \text{ implies that } c_2 = 0$$

Hence $c_1 \neq 0$, and the second boundary condition implies $J_0(\mu l) = 0$. On the other hand, it is clear that if $\mu > 0$, and $J_0(\mu l) = 0$ then $J_0(\mu x)$ is a solution of the given boundary problem.

Now we conclude that the set of non-zero characteristic values of the given problems is $\Lambda = \{\mu^2 : J_0(\mu l) = 0\}$, and for μ^2 in Λ the corresponding characteristic function is $J_0(\mu x)$. Note that, in this problem, $r(x) \equiv x$, so for any two distinctive characteristic values μ_1^2, μ_2^2 we have $\int_0^l xJ_0(\mu_1 x)J_0(\mu_2 x)\ dx = 0$. That is, if $\mu_1, \mu_2 > 0$, $\mu_1 \neq \mu_2$ and $J_0(\mu_1 l) = 0 = J_0(\mu_2 l)$, then $\int_0^l xJ_0(\mu_1 x)J_0(\mu_2 x)\ dx = 0$.

5.7. **Expansion of Arbitrary Functions in Series of Orthogonal Functions.**

31. Determine the coefficients in the representation $f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$, $(0 < x < \pi)$, in the following cases:

(a) $f(x) = 1$, (b) $f(x) = x$, (c) $f(x) = \begin{cases} 1 & (x < \pi/2), \\ \frac{1}{2} & (x = \pi/2), \\ 0 & (x > \pi/2) \end{cases}$

Solution. We know that $A_n \int_a^b r(x)[\varphi_n(x)]^2\ dx = \int_a^b r(x)f(x)\varphi_n(x)\ dx$ since the functions $\sin(nx)$, $n = 1, 2, 3, \ldots$ are orthogonal for $0 < x < \pi$.

(a) Thus

$$
\begin{aligned}
A_n \int_0^\pi [\sin(nx)]^2 dx &= \int_0^\pi \sin(nx)\ dx, \text{ and} \\
\int_0^\pi [\sin(nx)]^2 dx &= \frac{1}{2}\int_0^\pi [1-\cos(2nx)]\ dx \\
&= \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_0^\pi \\
&= \frac{\pi}{2}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\pi \sin(nx)\ dx &= \left[-\frac{1}{n}\cos(nx)\right]_0^\pi \\
&= \frac{1}{n}\left[1-\cos(n\pi)\right]. \\
\text{So } A_n &= \frac{1-\cos(n\pi)}{n\pi/2} = 2[1-\cos(n\pi)]/n\pi.
\end{aligned}
$$

(b)

$$
\begin{aligned}
A_n \int_0^\pi [\sin(nx)]^2 dx &= \int_0^\pi x\ \sin(nx)\ dx, \text{ and} \\
\int_0^\pi [\sin(nx)]^2 dx &= \frac{\pi}{2}, \text{ from (a), and} \\
\int_0^\pi x\sin(nx)\ dx &= \left[-\frac{1}{n}x\ \cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^\pi \\
&= -\frac{\pi}{n}\cos(n\pi).
\end{aligned}
$$

$$
\begin{aligned}
\text{So } A_n &= \frac{-\frac{\pi}{n}\cos(n\pi)}{\frac{\pi}{2}} \\
&= -\frac{2}{n}\cos(n\pi) = (-1)^{n+1}\cdot\frac{2}{n}
\end{aligned}
$$

(c)

$$\begin{aligned}
A_n \int_0^{\pi} [\sin(nx)]^2 dx &= \int_0^{\frac{\pi}{2}} \sin(nx)\ dx, \text{ and} \\
\int_0^{\pi} [\sin(nx)]^2 dx &= \frac{\pi}{2}, \text{ from (a), and} \\
\int_0^{\frac{\pi}{2}} \sin(nx)\ dx &= \left[-\frac{1}{n}\cos(nx)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right).
\end{aligned}$$

$$\begin{aligned}
\text{So } A_n &= \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)/\frac{\pi}{2} \\
&= \left(1-\cos\frac{n\pi}{2}\right)/\frac{n\pi}{2}.
\end{aligned}$$

32. (a-c) Show that the series obtained in Problem 31 cannot be differentiated term by term. (Investigate the convergence of the result in each case.)

Solution.
(a) $f(x) = 1$ and $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-\cos(n\pi)]\sin(n\pi x)$. Differentiating formally term by term, we get $f'(x) = 0$ and $f'(x) = \sum_{n=1}^{\infty} 2[1-\cos(n\pi)]\cos(n\pi x)$. So we're investigating

$$0 \stackrel{?}{=} \sum_{n=1}^{\infty} 2[1-\cos(n\pi)]\cos(n\pi x) \text{ in } (0,\pi).$$

Now for $0 < x < \pi$, as $n \to \infty$, $\cos(n\pi x) \not\to 0$ and for n odd, $2[1-\cos(n\pi)] = 4 \neq 0$. Thus, as $n \to \infty$, $2[1-\cos(n\pi)]\cos(n\pi x) \not\to 0$, so $\sum_{n=1}^{\infty} 2[1-\cos(n\pi)]\cos(n\pi x)$ diverges for all $0 < x < \pi$ and so indeed the series of Problem 31(a) cannot be differentiated term by term.

(b) $f(x) = x$ and $f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(n\pi x)$. Differentiating formally term by term, we get $f'(x) = 1$ and $f'(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot 2\pi\cos(n\pi x)$. So we're investigating

$$1 \stackrel{?}{=} \sum_{n=1}^{\infty}(-1)^{n+1}\cdot 2\pi\cos(n\pi x) \text{ in } (0,\pi).$$

Now for $0 < x < \pi$, as $n \to \infty$, $\cos(n\pi x) \not\to 0$ and certainly $2\pi(-1)^{n+1} \not\to 0$. Thus, as $n \to \infty$, $(-1)^{n+1}\cdot 2\pi\cos(n\pi x) \not\to 0$, so $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot 2\pi\cos(n\pi x)$ diverges for all $0 < x < \pi$ and so the series of Problem 31(b) cannot be differentiated term by term.

(c) Differentiating the series formally term by term, we get $f'(x) = 0$ and $f'(x) = \sum_{n=1}^{\infty} 2\left(1 - \cos\frac{n\pi}{2}\right)\cos(n\pi x)$. So we're investigating

$$0 \stackrel{?}{=} \sum_{n=1}^{\infty} 2\left(1 - \cos\frac{n\pi}{2}\right)\cos(n\pi x).$$

Now for $0 < x < \pi$, as $n \to \infty$, $\cos(n\pi x) \not\to 0$ and for $n = 4k+2$, $k = 0, 1, 2, \ldots$, $2(1 - \cos\frac{n\pi}{2}) = 4 \neq 0$. Thus, as $n \to \infty$, $2\left(1 - \cos\frac{n\pi}{2}\right)\cos(n\pi x) \not\to 0$, so $\sum_{n=1}^{\infty} 2\left(1 - \cos\frac{n\pi}{2}\right)\cos(n\pi x)$ diverges for all $0 < x < \pi$ and so, indeed, the series cannot be differentiated term by term.

33. (a) If the expansion $f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$ is valid in $(0, \pi)$, show formally that

$$\int_0^{\pi} [f(x)]^2\, dx = \frac{\pi}{2}\sum_{n=1}^{\infty} A_n^2.$$

(b) From this result, and from the results of Problems 31(a,b), deduce the following relations:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}, \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Solution. (a)

$$\begin{aligned} [f(x)]^2 &= \left(\sum_{n=1}^{\infty} A_n \sin(nx)\right)\left(\sum_{m=1}^{\infty} A_m \sin(mx)\right) \\ &= \sum_{n=1,m=1}^{\infty} A_n A_m \sin(nx)\,\sin(mx). \end{aligned}$$

So since $f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$ is valid in $(0, \pi)$,

$$\begin{aligned} \int_0^{\pi} [f(x)]^2\, dx &= \int_0^{\pi} \sum_{n=1,m=1}^{\infty} A_n A_m \sin nx \sin mx\, dx \\ &= \sum_{n=1,m=1}^{\infty} \int_0^{\pi} A_n A_m \sin(nx)\,\sin(mx)\, dx \\ &= \sum_{n=1,m=1}^{\infty} A_n A_m \int_0^{\pi} \sin(nx)\,\sin(mx)\, dx \end{aligned}$$

But $\int_0^{\pi} \sin(nx)\,\sin(mx)\, dx = 0$ if $m \neq n$. So

$$\begin{aligned} \int_0^{\pi} [f(x)]^2\, dx &= \sum_{n=1}^{\infty} A_n A_n \int_0^{\pi} [\sin(nx)]^2\, dx \\ &= \frac{\pi}{2}\sum_{n=1}^{\infty} A_n^2. \end{aligned}$$

(b) From Problem 31(a) $\Rightarrow 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-\cos(n\pi)\right]\sin(n\pi x),\ 0<x<\pi$. So

$$\begin{aligned}
\int_0^{\pi} 1^2\, dx = \pi &= \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left[1-\cos(n\pi)\right]\right)^2 \\
&= \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{4}{\pi^2}\cdot\frac{1}{n^2}\left(1-2\cos(n\pi)+\frac{1}{2}[1+\cos(2n\pi)]\right) \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[1-\cos(n\pi)]. \\
\text{So } \frac{\pi^2}{4} &= \sum_{n=1}^{\infty}\frac{1}{n^2}[1-\cos(n\pi)] \\
&= \sum_{m=0}^{\infty}\frac{2}{(2m+1)^2}, \qquad (n=2m+1) \\
\text{i.e. } \frac{\pi^2}{8} &= \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots
\end{aligned}$$

Similarly, from problem 31(b) $\Rightarrow x = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{2}{n}\sin(n\pi x),\ 0<x<\pi$. So

$$\begin{aligned}
\int_0^{\pi} x^2\, dx = \frac{\pi^3}{3} &= \frac{\pi}{2}\sum_{n=1}^{\infty}\left((-1)^{n+1}\cdot\frac{2}{n}\right)^2 \\
&= 2\pi\sum_{n=1}^{\infty}\frac{1}{n^2}. \\
\text{So } \frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots
\end{aligned}$$

34. Expand the function $f(x)=1$ in a series of the characteristic functions of the boundary-value problem $\frac{d^2y}{dx^2}+\lambda y=0$, $y(0)=0$, $ly'(l)+ky(l)=0$ $(k\geq 0)$, over the interval $(0,l)$.

Solution. Solving $\frac{d^2y}{dx^2} + \lambda y = 0$, we get

$$\begin{aligned} y(x) &= C_1 \sin(\sqrt{\lambda} \cdot x) + C_2 \cos(\sqrt{\lambda} \cdot x) \text{ and } y(0) = 0 \Rightarrow C_2 = 0 \\ \text{so } y(x) &= C_1 \sin(\sqrt{\lambda} \cdot x) \text{ and } y'(x) = C_1 \sqrt{\lambda} \cdot \cos(\sqrt{\lambda} \cdot x) \\ \text{and } ly'(l) + ky(l) = 0 &\Rightarrow l\sqrt{\lambda} \cdot C_1 \cos(\sqrt{\lambda} \cdot l) + kC_1 \sin(\sqrt{\lambda} \cdot l) = 0 \\ &\Rightarrow l\sqrt{\lambda} \cdot \cos(\sqrt{\lambda} \cdot l) + k \sin(\sqrt{\lambda} \cdot l) = 0 \\ &\Rightarrow \tan(\sqrt{\lambda} \cdot l) = -\frac{l\sqrt{\lambda}}{k} \\ &\textit{i.e.} \quad \tan(x) = -\frac{x}{k}, \ x > 0. \end{aligned}$$

So there are infinitely many solutions, one each on $\left[k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right]$ and let λ_k be the kth solution. Thus, the characteristic functions of this boundary-value problem are $\varphi_m(x) = \sin\left(\sqrt{\lambda_m} \cdot x\right)$, where $\sqrt{\lambda_m} > 0$.

This boundary-value problem is clearly Sturm-Liouville, and $r(x) = 1$ so the φ_m's are orthogonal. So

$$\begin{aligned} 1 = f(x) = \sum_{n=1}^{\infty} A_m \varphi_m(x) \text{ and} \\ A_m \cdot \int_0^l \sin^2(\sqrt{\lambda_m} \cdot x)\ dx &= = \int_0^l \sin(\sqrt{\lambda_m} \cdot x)\ dx \\ \text{So } \int_0^l \sin^2(\sqrt{\lambda_m} \cdot x)\ dx &= \frac{1}{2} \int_0^l \Big(1 - \cos(2\sqrt{\lambda_m} \cdot x)\Big)\ dx \\ &= \frac{1}{2}\left(l - \frac{1}{2\sqrt{\lambda_m}} \sin(2l\sqrt{\lambda_m})\right). \\ \text{and } \int_0^l \sin(\sqrt{\lambda_m} \cdot x)\ dx &= \frac{1}{\sqrt{\lambda_m}} \Big(1 - \cos(l\sqrt{\lambda_m})\Big) \end{aligned}$$

Thus,

$$A_m = \frac{\frac{1}{\sqrt{\lambda_m}}\left(1 - \cos(l\sqrt{\lambda_m})\right)}{\frac{1}{4\sqrt{\lambda_m}}\left(2l\sqrt{\lambda_m} - \sin(2l\sqrt{\lambda_m})\right)} = \frac{2\left(1 - \cos(l\sqrt{\lambda_m})\right)}{l\sqrt{\lambda_m} - \frac{1}{2}\sin(2l\sqrt{\lambda_m})}.$$