18.091 Lab 2
Rates of Convergence

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1 Introduction

In this experiment we investigated the correlation between the rate at which seeds upon iteration converge to a fixed point of a given iterating function and the absolute value of the derivative of this iterating function at the fixed point. The fixed point of each function is either attracting or neutral and weakly attracting. The iterating functions we studied were

1. $F[x] = x^2 + 0.25$,

2. $F[x] = x^2$,

3. $F[x] = x^2 - 0.24$,

4. $F[x] = x^2 - 0.75$,

5. $F[x] = 0.4 \cdot x \cdot (1 - x)$,

6. $F[x] = x \cdot (1 - x)$,

7. $F[x] = 1.6 \cdot x \cdot (1 - x)$,

8. $F[x] = 2 \cdot x \cdot (1 - x)$,

9. $F[x] = 2.4 \cdot x \cdot (1 - x)$,

10. $F[x] = 3 \cdot x \cdot (1 - x)$,

11. $F[x] = 0.4 \cdot Sin(x)$,

12. $F[x] = Sin(x)$.

2 Methods

For each function we measured the rate of convergence of a single seed $x_0 = 0.2$ by counting the number of iterations it took to get within ± 0.001 from the fixed point p.

3   Data and Observations

On the following table we list for each function F its corresponding number from the list in the introduction, the unique attracting or weakly attracting fixed point p, the absolute value of the derivative of F evaluated at p, the classification of the fixed point p (attracting or neutral), and the number of iterations it took for the seed to converge.

F No.	p	$|F'(p)|$	Classification of p	# Iterations for convergence
1	0.5	1	Neutral	991
2	0	0	Attracting	3
3	-0.2	0.4	Attracting	1
4	-0.5	1	Neutral	$499,479$
5	0	0.4	Attracting	6
6	0	1	Neutral	990
7	0.375	0.4	Attracting	8
8	1/2	0	Attracting	4
9	0.583333	0.399998	Attracting	5
10	2/3	1	Neutral	$55,374$
11	0	0.4	Attracting	6
12	0	1	Neutral	$2,999,919$

For the functions F for which $F'(p) = 1$, we also calculated the second and third derivatives at p and recorded those values on the following table:

F No.	$F'(p)$	$F''(p)$	$F'''(p)$	# Iterations for convergence
1	1	2	0	991
4	-1			
6	1	-2	0	990
10	-1			
12	1	0	-1	$2,999,919$

4   Discussion

There is a clear correlation between the classification of the fixed points p and the number of iterations needed for convergence of the seed. Namely, for all of the attracting points p the seed required eight or less iterations to converge, while for all of the neutral points p which are weakly attracting the seed required 990 iterations or more to approach the fixed point p.

Note that other relations between the seed and the fixed point, including the distance between the seed and the fixed point, the family of functions studied, and the absolute value of the derivative when it is different from 1, all seemed to be independent of the rate of convergence we studied.

There are only three functions (numbers 1, 6 and 10) which have their derivatives evaluated at p equal to 1. For these we can note two interesting correlations. First, as the absolute value of the second derivative of F at p gets smaller the number of iterations required for the seed to converge to p increases. Second, as the absolute value of the third derivative of F at p gets smaller the number of iterations required for the seed to converge to p decreases.

The experiment led us to the conclusion that there is a well-defined relation between the classification of the fixed points and the rate of convergence of a seed towards the fixed points, namely, that attracting fixed points attract seeds much faster than neutral points which are weakly attracting. However, it seems that other properties of the iterating function, the fixed point, or the seed, are not related to the rate of convergence we studied.