The Quadratic Family and the Cantor Set

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1 Quadratic Family: $Q_c(x) = x^2 + c$

We have already begun to investigate the behavior of this family of functions, particularly in the first lab we did. For many values of c, the behavior of the function is not particularly interesting, but we are able to analyze the conditions leading to any particular type of behavior. Our concern in this chapter and this lecture will be other values of c, particularly those for $c \leq -2$. The fixed points of the quadratic map for an arbitrary value of c are given by the solutions to $Q_c(x) = x$, that is $x^2 - x + c = 0 \Longrightarrow p_\pm = \frac{1}{2}(1 \pm \sqrt{1-4c})$. Using this and the first derivative $Q_c'(x) = 2x$, we know the following things about the quadratic map:

1. For $c > 1/4$, all orbits go to infinity. For $c = 1/4$, Q_c has a single neutral fixed point at $1/2$, for this is where the discriminant is zero.

2. When $c < 1/4$, the graph intersects the line $y = x$ in two places, and we have two fixed points given as above, p_+ and p_-. p_+ is always repelling since $Q_c'(p_+) = 1 + \sqrt{1-4c} > 1$ for $c < 1/4$, but since $Q_c'(p_-) = 1 - \sqrt{1-4c}$, we have to determine where $Q_c'(p_-)$ has magnitude greater than 1 and less than 1. A simple analysis shows that $|Q_c'| < 1$ for $-3/4 < c < 1/4$, $|Q_c'| = 1$ for $c = -3/4$, and $|Q_c'| > 1$ for $c < -3/4$, so these ranges lead to an attracting, neutral, and repelling p_- respectively. This is a basic bifurcation.

We are going to concern ourselves with what happens when c gets smaller, particularly when the minimum of Q_c reaches the line $y = -2$ and goes below it. We investigated in Experiment 3.6 the behavior for $c = -2$ and some $c < -2$, but now we are going to establish a more analytic basis for what we're talking about.

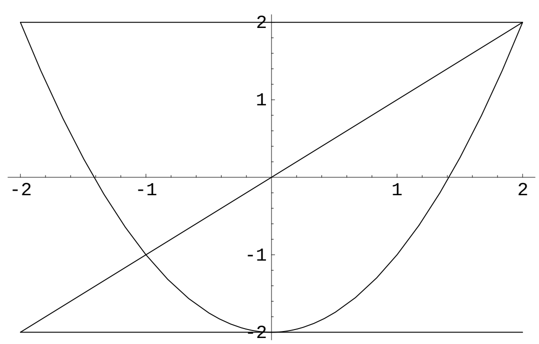

Figure 1: $y = x^2 - 2$ (horizontal lines added)

1.1 Case 1: $c = -2$

From experimental observations Q_{-2} behaved chaotically for the essentially all of the seeds we picked for orbits. One of the very obvious things is that when we choose a seed that is not between the fixed points, we have an orbit that tends to infinity very quickly, since the fixed points are repelling as above. When $x_0 \in [-2, 2]$ the orbit is confined to the 4-by-4 box shown above, which can be seen by graphical intuition, in addition to drawing orbit diagrams. It turns out that the function's behavior is not completely chaotic, and to analyze its characteristics we note some of the mapping properties of Q_{-2} and its iterates.

Firstly, as we see from the graph of Q_{-2}, the function maps both $[-2, 0]$ and $[0, 2]$ *onto* $[-2, 2]$, so that for each $x \in [-2, 2]$, there are two points y in the whole interval for which $Q_{-2}(y) = x$. Similarly, since the image of $[0, 2]$ under Q_{-2} is $[-2, 2]$, the second iterate produces two images of $[-2, 2]$ from $[0, 2]$, and similarly for $[-2, 0]$ under the second iterate. This follows also from composition of functions, since Q^2_{-2} is really $Q \circ Q$. Continuing with this logic, we note that there will be 2^n disjoint intervals of $[-2, 2]$ which get mapped to $[-2, 2]$ by $Q^n_{-2} = Q \circ Q \circ \cdots \circ Q$ (n times). What is really crucial here is that the line $y = x$ will cross the graph of Q^n 2^n times! The important question now is, what do the fixed points of Q^n_{-2} tell us? We know this: they are n-cycles.

The really fascinating part of this development is that we have shown that a seemingly chaotic function has quite a lot of periodic points, which we can find and graph. For example $Q^2_{-2}(x) = x$ leads to two 1-cycles we already know (1-cycles are 2-cycles of course), and two new 2-cycles, $x = \frac{1}{2}(-1 \pm \sqrt{5})$. The following graph gives you a picture of the 8 periodic points we can gather from the 3rd iterate.

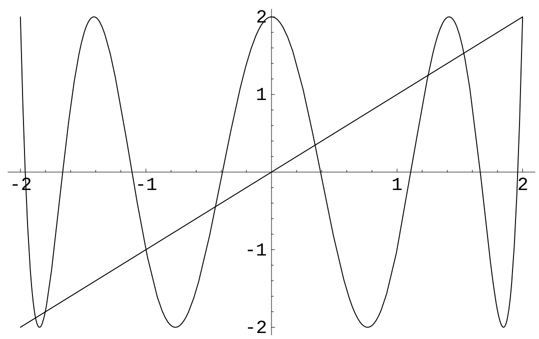

Figure 2: $y = Q^3_{-2}(x)$

1.2 Case 2: $c < -2$

As we have noted, when an orbit leaves the interval defined by the fixed points of $Q_c(x)$, the orbit tends to infinity. When $c < -2$, this seems to always be the case, as in our experimentations. But we are going to examine more closely the conditions under which an orbit leaves the interval $[-2, 2]$. When $c < -2$, the minimum of Q_c dips below the line y = 2, leaving the 2x2 "box" and tending to infinity. We want to identify which intervals in $[0, 1]$ will have an orbit which finds it way outside of the 2x2 box. There will be an initial interval corresponding to the points in which Q_c intersects $y = -2$. Call this interval A_1. For each $x \in A_1$ (and in the whole interval for that matter), $Q_c^{-1}(x)$ consists of two points. Thus, if we define A_2 to be the intervals which hit A_1 after 1 iterate, we note that A_2 contains two intervals. At each step, the next A_k will have twice the number of intervals as A_{k-1}, since each interval in A_{k-1} has two *preimages*.

The result of all this is to take

$$\Lambda = [-2, 2] - \bigcup_{1}^{\infty} A_k.$$

These are the points of $[0, 1]$ that stay in the interval for every iterate. This reduced set contains some points since the fixed points don't go anywhere. We note that at each step the length of intervals removed is going to be decreasing. The idea behind this construction is what leads to the idea of a Cantor set.

2 Cantor Sets

For many undergraduates, the Cantor "middle-thirds" set is the first, let us say, interesting set they see constructed. With respect to chaos theory, it is a good example of a simple fractal. It also lends itself to some interesting counterexamples in measure theory, among other things.

The construction is pretty easy to visualize. First, we start off with the closed interval $[0,1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. For each remaining interval in the set, remove the open middle third of it. We continue to do this *ad infinitum*, step by step, ending up with a set C. Making a drawing is really the best way to visualize it.

2.1 Ternary Expansions

In order to discuss the Cantor set, we must briefly introduce *ternary* expansions. These are analogous to binary and decimal expansions, but as one might guess, we work in base 3 versus base 2 or 10. For a ternary number $0.a_1a_2\ldots$, this is equivalent to:

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

We will present here a very useful division algorithm for determining expansions in alternate bases. It is especially useful for working out expansions very quickly. Like any algorithm, this one has theoretical justification which we will present as well. It is this:

1. Write the numerator of the rational x you're finding the ternary expansion of on the first line, and a period to its right to designate the start of the ternary expansion.
2. General step: Multiply the number on the previous row by 3, and subtract the largest $a \cdot b$ from it, where b is the denominator of x. This a is the ith ternary place.
3. Continue in this way until a 0 is reached, or the number being multiplied repeats.

This is best elucidated by example.

Example: $x = 31/121$

	31/121	
31	.	
93	0	
279	2	$279 - 2 \cdot 121 = 37$
111	0	
333	2	$333 - 2 \cdot 121 = 91$
273	2	$273 - 2 \cdot 121 = 31$
93	0	repitend

So the conclusion is that $31/121 = .\overline{02022}$. The justification for this is as follows. If we want to know how many times $1/3$ goes into a/b, then

$$\frac{a}{b} - \frac{n}{3} = \frac{3a}{3b} - \frac{nb}{3b} = \frac{3a - nb}{3b}.$$

We choose the largest n (which could easily be 0) for which $3a - nb$ is still nonnegative. We proceed with the new positive rational numbers created by this process. For the general step, determining how many times $1/3^k$ goes into $a/3^{k-1}b$, then as before:

$$\frac{a}{3^{k-1}b} - \frac{n}{3^k} = \frac{3a - nb}{3^k b}.$$

Again the largest n is chosen so that the resulting rational number is nonnegative. This justifies the algorithm.

Proposition 1. *If $x \in C$, then x has a ternary expansion consisting of only 0's and 2's.*

Proof. We prove this statement by induction. It is clear for the first ternary place, since the removed segment $(\frac{1}{3}, \frac{2}{3})$ corresponds to those $x \in [0,1]$ of the form $1/3 + \sum_{i=2}^{\infty} a_i/3^i$. Let us assume it is true for all ternary places of x up the kth place, and we have not yet removed the kth set of "middle-thirds" in constructing the Cantor set. Then x lies in some segment in C, and if x lies in the middle third of that segment, then x is not in C, thus the kth ternary place of x must be a 0 or a 2. The converse of this proposition is proven to be true in the same way. $\square$

Proposition 2. *The Cantor set is uncountable.*

Proof. Rather than present the (somewhat careless) proof in Devaney's text, we use a simple diagonalization argument. First, suppose there existed a bijection $f : \mathbb{N} \longrightarrow C$ (that is, C is countable). Then we can enumerate the elements of $C = \{c_1, c_2, \dots\}$. Let c_{ik} denote the kth number in the ternary expansion of c_i. Define a new ternary number x by the rule $x_k \equiv 2 - c_{kk}$, with $x = 0.x_1x_2x_3\dots$. Then the ternary expansion of each x differs from that of each $c \in C$ in at least one place, but since x has a ternary expansion consisting of only 0's and 2's, $x \in C$, contradicting the assumption that f was a bijection. $\square$